A Theory of Sovereign Bond Safety:
Country Size and Equity Rebalancing*

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Abstract

This paper provides a theoretical framework to understand sovereign bond safety in both normal and in crisis times. Using a continuous-time two-country Lucas tree model with equity constraint, we show that the country-size effect and the equity-rebalancing effect are the key determinants of sovereign bond safety. The country size effect spills over home production risk to a smaller country through trade and equity rebalancing; equity constraint limits equity rebalancing and creates endogenous uncovered interest parity (UIP) deviations in both normal and crisis times. In the period of crisis, the larger country’s sovereign bond becomes a global safe asset when the country size effect dominates the equity rebalancing effect, as is the case with the United States. Our model mechanisms qualitatively explain the empirical evidence on the country-size and equity-rebalancing effect for both the G10 and emerging market currencies. Our model predictions also reconcile with the empirical facts of flight-to-safety and the covered interest parity (CIP) in both normal and crisis times.

JEL Classification: F31, F41, G15.

Keywords: Uncovered interest parity (UIP); safe assets; sovereign bonds; equity rebalancing; continuous-time finance.

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1 Introduction

Sovereign bonds valued as safe assets by global investors pay lower expected returns that can not be compensated by exchange rates movements. Such persistent difference in sovereign bond return, reflecting variations in sovereign bond safety, is also known as the failure of the *uncovered interest rate parity* (UIP). The failure of UIP to hold in data has been a long-standing puzzle in International Finance since the pioneering work of Fama (1984). Moreover, recent literature has documented that the UIP premium reverses sign\(^1\) and that the reversal seems to be systematically correlated with the period of crisis (Corsetti and Marin (2020)) and the global risk appetite (Kalemli-Ozcan and Varela (2021)).

What determines sovereign bonds safety, reflected by their relative returns (UIP premium), in both crisis and normal times? Sovereign bond safety is at the core of international macroeconomics and finance, yet there’s no unifying theory that jointly explains the sovereign bonds safety both in normal and crisis times, especially through the dynamics in equity markets. Both the international bonds market and equity markets experienced dramatic fall during the period of crisis. However, the literature has either focused on the bond markets to study the exchange rate dynamics while leaving the equity markets unattended (Gabaix and Maggiori (2015) Itskhoki and Mukhin (2021)), or on the portfolio rebalancing dynamics in the equity markets solely (Hau and Rey (2008); Camanho, Hau, and Rey (2022)).

This paper leverages the insights on portfolio rebalancing from the equity markets to generate rich dynamics in the foreign exchange and the sovereign bonds market. Our theory shows that the relative size of the country as well as the equity rebalancing channel jointly determine sovereign bond safety. Using a two-country Lucas tree model with equity constraints, we characterize the model mechanism in closed-form and reconcile the observed UIP patterns both in normal and in crisis times. We propose that the interaction between *country-size effect* and the *equity-rebalancing effect* due to equity constraint is the key driver of UIP patterns. Our model mechanisms can qualitatively explain the pattern of UIP reversals combined with the country-size and equity-rebalancing effect in the period of financial crisis of 2008 for both the G10 and emerging market economy (EME) currencies.

In normal times, the two countries in our model – a smaller home country and a larger foreign country – can perfectly share consumption risks through freely adjusting their equity and bond holdings. The *country-size effect* makes the larger country’s bond a global safe asset in normal times as the larger country constitutes most of total world consumption risks through the international trade and financial market. Therefore, investors are willing to pay a safety premium for the larger country bond by receiving lower expected returns. This rationalizes the observed UIP premium

\(^1\)Recent literature documents for advanced economies, high interest rate currencies have higher expected returns over the near future and the UIP reverses sign after about eight quarters (Engel (2016); Dahlquist et al. (2023)).
during normal times. Throughout the paper, we treat the U.S. as the larger foreign country, and the G10- or EME-currency countries as the smaller home country.

In the period of crisis, both the home and foreign countries are constrained in their equity holdings and have to take on more home risks in consumption than they would ideally prefer. However, the two model mechanisms – namely the country-size and equity-rebalancing effect – work differently for the smaller home and the larger foreign country in crisis. For investors in the smaller home country, the equity-rebalancing effect competes with the country-size effect. If the country-size effect dominates, home bond becomes safer for home investors in crisis and UIP reversal occurs; if the equity-rebalancing effect dominates, the larger foreign country’s bond (U.S.) remains safer for home investors and the flight-to-safety occurs. For investors in the foreign larger country, the equity-rebalancing effect collaborates with the country-size effect and the safety of the larger country’s bond is strengthened for home investors in the period of crisis.

In our model, the equity constraint is the key and only departure from an otherwise standard two-country Lucas tree model as in Cochrane, Longstaff, and Santa-Clara (2007). Each country has to hold at least a fraction of their domestic equity - they can not issue as much domestic equity share as they would like to. The equity constraints deliver the equity home bias well-documented in the literature (Hau and Rey (2008); Coeurdacier and Rey (2013)). In this paper, we argue that shocks that tightens the equity constraint facing home country drives the system into crisis. That is, the maximum holding of foreign equity by home country decreases during crisis. The equity constraints in our model fall into the balance sheet constraints class and is supported by empirical evidence (Du, Tepper, and Verdelhan (2018b); Du, Hébert, and Huber (2019)).

Both the country-size spillover and equity-rebalancing effect during the financial crisis are well-founded by empirical evidence. On the country-size effect, we found that the relative size of G10 currencies relative to the U.S. peaks at the 2008 financial crisis, consistent with the strong and negative correlation on the UIP premium and country size on the advanced countries as documented by Hassan (2013). By comparison, we didn’t find such correlation for the EME-currencies, suggesting that the country-size effect might not the driver for UIP reversals for the EMEs. On the equity-rebalancing effect, we found that the effect is present for both the G10 and EMEs with the larger foreign country (U.S.) holdings of foreign assets increased during the period of financial crisis, while at the same time the home country (G10 or EME) holdings of home assets have no sizable change. Our empirical evidence on equity rebalancing is consistent with the increase in equity home bias during the financial crisis (Wynter (2019); Atkeson, Heathcote, and Perri (2022)).

Our model predictions also reconcile with the empirical facts on deviations from the covered interest rate parity (CIP) and convenient yields. The failure of CIP implies a breakdown of the

2For example, Du, Tepper, and Verdelhan (2018b) uses banking regulation to test the balance sheet constraints and shows that the balance sheet constraints have impact on asset prices. Du, Hébert, and Huber (2019) provides direct evidence that the risk of balance sheet constraints becoming tighter is priced.
no-arbitrage condition, contrasts the friction-less market assumption, and points to models with financial frictions. There is market segmentation during crisis time in our model with equity constraints. Such market segmentation limits arbitrage across markets and law of one price is violated in crisis time, which generates the deviations from CIP.

1.1 Literature review

This paper builds on and contributes to the study of currency risk premia, safe asset determination, and portfolio rebalancing.

First, our study on currency risk premia is related to the discussion of safe asset, UIP puzzles, and exchange rate risk hedging. Gopinath and Stein (2020) shows that a currency that hedges exchange rate risk endogenously has lower return and becomes the dominant currency due to the complementarity in trade invoicing and banking, taken exchange rates as exogenous in the model. Gabaix and Maggiori (2015) studies exchange rate determination in a segmented international financial market with global financiers facing credit constraints and explains the UIP violation, taken bond demand as given. Itskhoki and Mukhin (2021) also looks at a segmented international financial market and introduce exogenous noise trader shocks into a standard international real business cycle model to account for the UIP violation puzzle. In our model, exchange rates, the demand for bonds as well as equities are all endogenously determined.

Second, our study on sovereign bond safety contributes to the existing literature by jointly explaining bond safety in both normal and crisis times. The literature proposed various fundamental determinants of bond safety but cannot The documented determinants include coordination of investors (He, Krishnamurthy, and Milbradt (2019)), financial depth (Maggiori (2017)), heterogeneous risk aversion coefficients (Gourinchas and Rey (2007)), rare disaster and heterogeneous disaster resilience (Farhi and Gabaix (2016), Corsetti and Marin (2020)), and country size effect solely (Hassan (2013), Martin (2011)). While each of the existing theory can explain only one of the empirical facts mentioned above, our paper jointly explain UIP violation in both normal and crisis times, as well as the facts on the flight to safety, CIP deviations and convenience yields in the period of crisis.

In addition, our two-country Lucas-tree model builds on classic continuous-time asset pricing framework. Starting from fiction-less models: Cochrane, Longstaff, and Santa-Clara (2007) solves a two-tree model with perfect substitutable goods and leaves no space for exchange rates. Pavlova and Rigobon (2007) solves a two-tree model with log-linear preference, where the country size spillover effect does not show up as a result of the knife-edge case of CES consumption. Martin (2011) solves the price levels in a general two-trees model with and shocks following any Levy process whereas we instead focus on optimal portfolio trade-off and solve for intertemporal risk pricing and Euler equations. Continuing to models with financial frictions: Pavlova and Rigobon (2008)
builds a center-periphery three-country model with exogenous country size parameters and general portfolio constraints to study contagion and exchange rate movements in crisis. Garleanu and Pedersen (2011) shows that deviations from law of one price emerges in a heterogeneous risk-averse agents model with linear margin constraints.

Lastly, the portfolio rebalancing literature has either focused on the bond markets to study the exchange rate dynamics while leaving the equity markets unattended (Gabaix and Maggiori (2015); Itskhoki and Mukhin (2021)), or on the portfolio rebalancing dynamics in the equity markets solely (Hau and Rey (2004); Camanho et al. (2022)). This paper bridges this gap.

1.2 Outline

The rest of the paper is organized as follows. Section 2 presents the empirical facts on UIP reversal, country-size effect and the equity-rebalancing effect during the financial crisis. Section 3 introduces and set up the model. Section 4 discusses the model mechanisms and predictions in the complete market setting with no equity constraints. Section 5 introduces the equity constraints and addresses the model solutions in both normal and crisis times. Section 6 explains how the model predictions reconcile with the empirical facts on UIP reversal, flight-to-safety and CIP. Section 7 considers model solutions when there are changes in the equity constraint and trade elasticity due to financial development. The last section concludes.

2 Motivating Empirical Facts

2.1 Data Description

We combine a few public databases to provide empirical evidence on the pattern of UIP premium and its relation with country size and the equity portfolio rebalancings at the financial crisis. The data on exchange rates and government bond yields used to construct UIP premium are from the Bank for International Settlements (BIS). The data on nominal GDP are retrieved from the International Financial Statistics (IFS) provided by the International Monetary Fund (IMF).

We use both the Coordinated Portfolio Investment Survey (CPIS) provided by the IMF and the market capitalization database from the World Bank to compute the equity portfolio holdings measure. The CPIS dataset has a wide coverage of countries but it only reports cross-border demand and not demand for domestic equities held by domestic investors. We therefore follow Koijen and Yogo (2020) and use the total market capitalization of all domestic listed firms reported by the World Bank as the total supply of domestic equities. We then use the total foreign demand aggregated from the CPIS data to subtract from the total supply of domestic equities to back out the holdings of domestic equities.

Our sample includes both the group of G10 currencies and the group for emerging market
currencies. The G10 currencies besides the U.S. Dollar (USD) are Euro (EUR), Pound Sterling (GBP), Japanese Yen (JPY), Australian Dollar (AUD), New Zealand Dollar (NZD), Canadian Dollar (CAD), Swiss Franc (CHF), Norwegian Krone (NOK) and the Swedish Krona (SEK). We have 12 emerging market currencies in our sample chosen for their availability in the exchange rates, government bond yields and equity holdings data. These 12 currencies are Brazilian Real (BRL), Chilean Peso (CLP), Colombian Peso (COP), Hungarian Forint (HUF), Indonesian Rupiah (IDR), Indian Rupee (INR), Mexican Peso (MXN), Malaysian Ringgit (MYR), Philippine Peso (PHP), Russian Ruble (RUB), Thai Baht (THB) and South African Rand (ZAR).

2.2 Definition on UIP premium

To fix ideas, let us define UIP premium as the excess return of home currency asset against the U.S. Dollar (foreign currency). The home currency can be any G10 currencies other than the USD or an EME currency. The UIP premium in log points is therefore:

\[ \lambda_t \equiv (i_t - i_t^{US}) - (E_t s_{t+h} - s_t) \]  

(1)

where \( i_t \) and \( i_t^{US} \) are local and U.S. annualized one-year government bond yields; \( h \) is the 12-month horizon. Exchange rate \( s_t \) is in units of local currency per USD; an increase in \( s_t \) would imply local currency depreciation against the USD. When \( E_t \lambda_{t+h} = 0 \), the UIP condition holds and there’s no excess return from the currency carry trade. If \( E_t \lambda_{t+h} > 0 \), there’s positive excess returns for the currency trade that longs home currencies and shorts the USD; vice versa for \( E_t \lambda_{t+h} < 0 \). In the data, we use realized exchange rates to measure UIP premium due to the lack of survey data on exchange rates expectation.

2.3 Three Stylized Empirical Facts

We highlight three stylized facts linking reversals in UIP premiums, the country-size effects and the equity rebalancing channel at the financial crisis for both the G10 and EME currency group.

**Stylized Fact I:** UIP premium – defined as the excess returns in local currencies against the USD – reverses sign during the financial crisis for both the G10 and EME currencies.
Our work is motivated by the empirical facts on UIP reversals in the period of crisis, as shown in Figure 1. The pattern of UIP reversals during the the period of crisis is robust for both the G10 and the emerging market economies. Average realized local-currency premium falls dramatically and turn negative in the start of the great financial crisis of 2008/09 before reverting back to positive. A negative local-currency excess return implies that investors are losing profits investing in local currencies against the USD. We also plot UIP premium at the currency level and found that most currencies share the same feature of UIP reversal during the crisis, as reported in Table B.1 and B.2 in the Appendix.

While we use the 2008 great financial crisis for illustration in Figure 1, the empirical facts on UIP reversals in crisis times are not unique to the great financial crisis, as verified in several recent papers. For example, Kalemli-Ozcan and Varela (2021) shows that the realized UIP premium in both the advanced and the emerging market economies correlate strongly with the VIX index, a proxy for global risk perception that has been widely used in the international finance literature (see for example, Rey (2015)). Specifically, the correlation between UIP premium and VIX is positive and highly statistically significant, suggesting that higher global risk associates with higher UIP premia in local currencies against the US Dollar. In addition, Corsetti and Marin (2020) documented systematic UIP reversals for the carry trade in the Pound Sterling (GBP) against the US. Dollar that co-move with the episodes of crises periods in the US.

Stylized Fact II: The relative country size for the G10 currency group peaks during the financial crisis.

Note that while several recent papers, including Kalemli-Ozcan and Varela (2021) and Dahlquist et al. (2023), look at the reversal of local-currency UIP premium from the negative to the positive sign, we focus on the reversal of UIP premium that turns negative in the period of crisis.
Figure 2: Relative GDP Ratio for G10-currency and EME Group

We construct the relative country size of the G10-currency (or EME) group to the world as the weighted average nominal GDP of G10 over its sum with the US of the same year. We treat G10-currency (or EME) as the home country, and the U.S. as the foreign and the larger country. Therefore, the share of country size constructed is a proxy for the relative country size in the world. Figure 2 presents the time-series on this relative country size and shows that the size for G10-currency group peaks at the great financial crisis of 2008, with an increase of more than 3% in the 2008 financial crisis. In comparison, the EME group relative size was rather flat with no visible change during the 2008 financial crisis and only peaks three years after. Table B.7 in the appendix gives the time series for country-specific relative size.

The finding that the relative country-size for G10-currencies, but not EMEs, peaks at the financial crisis of 2008 suggests that the country-size effect might not be the driver for the UIP reversal in the EMEs. To confirm this, we preform OLS regressions for annualized UIP premium on the country-specific size relative to the world, as reported in Table B.3 and B.4. The relation between the UIP premium and relative country size is negative and strongly significant for both the full sample and the sub-sample using only years before and after the 2008 financial crisis. This suggests that the increase in the relative country-size of G10 in the 2008 financial crisis can potentially explain the fact on UIP reversal addressed above. In comparison, the OLS results for EMEs are largely inconclusive, with the full sample OLS coefficient producing the wrong sign on the relation between UIP premium and GDP and the sub-sample results largely insignificant.

We also verify the relation between the UIP premium and the within-group country size for the G10 and EME group separately. Consistent with Hassan (2013), we found negative and statistically significant correlation between the UIP premium and the within-G10 country size share for the
advanced economies (our G10 currency group). However, the relation between UIP and country size breaks down for the group of emerging market economies. Scatter plots in Table B.8 attest to this claim. The OLS regression with year and country fixed effects also confirms the statistically significant relation for the G10-currency group, as reported in Table B.5, and the insignificant relation for the EME group, as reported by B.6.

Stylized Fact III: Equity re-balancing effects are present for both the G10- and EME-currency group in the 2008 financial crisis, with an increase in the foreign (US) investor holdings of foreign equities ($\chi_{F,F}$) and no change in the home (G10 or EME) investor holdings of home equities ($\chi_{H,H}$).

We provide four measures on cross-border equities holdings to gauge the effects on equity rebalancing channel during the financial crisis. All measures treat G10(or EME) currencies other than the USD as home and the USD as foreign. We define $\chi_{H,H}$ as the share of home investors’ holdings of home equities over the total supply of home equities; $\chi_{F,H}$ is foreign investors’ holdings of home equities over the total supply of home equities; $\chi_{H,F}$ is home investors’ holdings of foreign equities over the total supply of foreign equities; Finally, $\chi_{F,F}$ is foreign investors’ holdings of foreign equities over the total supply of foreign equities. By construction, $\chi_{H,H}$ and $\chi_{F,H}$ add up to 1 as they share the total supply of home equities as the same denominator; $\chi_{H,F}$ and $\chi_{F,F}$ also add up to 1 and both use the total supply of foreign (USD) equities as the denominator.

Figure 3 presents the time series of these four measures on equity holdings. Consistent with the literature that documents the phenomenon on equity home bias for the U.S as the global safe assets (Atkeson et al. (2022)), we find that the share of holdings of US equities of by US investors (denoted by $\chi_{F,F}$, as USD is foreign) increases during the 2008 financial crisis. The finding is robust for using both the G10-currency group and EME-currency as home. By construction, the home investor holdings of US equities ($\chi_{H,F}$) dropped at the financial crisis. At the same time, there’s no sizable change in the share of holdings on home (G10 or EME) equities during the 2008 financial crisis, as shown by the graphs on $\chi_{H,H}$ and $\chi_{F,H}$.

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4 Note that the share of equity holdings needs to be distinguished from the equity portfolio shares (denoted by $\theta$) that will be introduced later in the model section of the paper. For example, we use $\theta_{H,H}$ to define the share of home equities held by home investors over the total wealth of home investors.
Figure 3: Equity Shares for G10- and EME-currency group

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\end{figure}

**Note:** This panel of figures present the average share of equity holdings or the G10 (top, blue) and the EME group (bottom, red). For all figures, home (H) is G10 or EME country, and foreign (F) is the U.S. We use the weighted average across all country holdings in the G10-currency group for computing the equity share for G10 (top); we use the simple average across EME-currency group for computing the equity share in the EME group (bottom) as we don’t have a fully balanced panel. The equity share $\chi_{H,H}$ is defined as home investors’ holdings of home equities over the total supply of home equities; $\chi_{F,H}$ is foreign investors’ holdings of home equities over the total supply of home equities; $\chi_{H,F}$ is home investors’ holdings of foreign equities over the total supply of foreign equities; and $\chi_{F,F}$ is foreign investors’ holdings of foreign equities over the total supply of foreign equities. The shaded line indicates the 2008 financial crisis.
3 Model Set-up

Time is continuous and infinite horizon, \( t \in [0, +\infty) \). There are two countries in the world, home country (denoted by \( H \)) and foreign country (denoted by \( F \)). For ease of illustration, we will call home country the UK (one of G10-currency countries) and foreign country the US.

**Technology**  Each country is endowed with a tree producing domestic good. The two trees evolve as follows,

\[
\begin{align*}
\frac{dY_{H,t}}{Y_{H,t}} &= \mu_{H,t} dt + \sigma_{H,t} dZ_{H,t} \\
\frac{dY_{F,t}}{Y_{F,t}} &= \mu_{F,t} dt + \sigma_{F,t} dZ_{F,t}
\end{align*}
\]

where \( \{\mu_{H,t}, \mu_{F,t}, \sigma_{H,t}, \sigma_{F,t}\} \) are exogenous parameters (or processes). For simplicity, we assume throughout the paper that \( \mu_{H,t} = \mu_{F,t} = \mu \) and \( \sigma_{H,t} = \sigma_{F,t} = \sigma \).

**Preferences**  In order to highlight our mechanism, we assume homogeneous preference, logarithmic utility, and no consumption home bias for the representative agents of the two countries. The final consumption is a CES aggregate of the two goods produced by the two countries. The expected utility of the representative agent in country \( i \), takes the form

\[
\mathbb{E} \int_0^\infty e^{-\rho t} \log C_{H,t} \, dt
\]

where

\[
C_{H,t} = \left[ a \frac{1}{\eta} Y_{H,t}^{\frac{\eta - 1}{\eta}} + (1 - a) \frac{1}{\eta} Y_{F,t}^{\frac{\eta - 1}{\eta}} \right]^{\frac{\eta}{\eta - 1}}
\]

\( a \) is the share parameter \(^5\). \( \eta \) is the elasticity of substitution between the two goods, assumed to be greater than 1 and smaller than infinity.

**Numeraire**  Define 1 unit of the CES basket of total output \( \bar{Y}_t \) as numeraire throughout the paper,

\[
\bar{Y}_t = \left[ a \frac{1}{\eta} Y_{H,t}^{\frac{\eta - 1}{\eta}} + (1 - a) \frac{1}{\eta} Y_{F,t}^{\frac{\eta - 1}{\eta}} \right]^{\frac{\eta}{\eta - 1}}
\]

Denote the process of total output \( \bar{Y}_t \) as

\[
\frac{d\bar{Y}_t}{\bar{Y}_t} = \bar{\mu}_t dt + \bar{\sigma}_t dZ_t
\]

\(^5\)Unlike Pavlova and Rigobon (2008) where \( a \) represents the country size, here in our model \( a \) is not a key parameter of interest.
where \( dZ_t = [dZ_{H,t}, dZ_{F,t}]^T \).

**International trade market and exchange rate**  The international trade market (of home and foreign goods) is frictionless. Denote \( p_{t}^{H} \) as the price of home good and \( p_{t}^{F} \) as the price of foreign good. The real exchange rate is given by the relative price of home and foreign good\(^6\),

\[
e_t \equiv \frac{p_{t}^{H}}{p_{t}^{F}}
\]

and \( e_t \) is also the terms of trade in this model. And denote the endogenous process of real exchange rate, \( e_t \), as

\[
\frac{de_t}{e_t} = \mu^e_t dt + (\sigma^e_t)^T dZ_t
\]

**Equity**  Each country can issue domestic equity shares in unit supply. The equities are risky claims to domestic trees. Denote \( S_{t}^{H} \) and \( S_{t}^{F} \) as the total value of domestic equity and foreign equity respectively. Define \( \chi_{t}^{H,H} \) as the the share of home stock market (apple tree) held by home investor, \( \chi_{t}^{H,F} \) as the share of foreign stock market (orange tree) held by home investor. And similarly define \( \chi_{t}^{F,H} \) as the share of home stock market (apple tree) held by foreign investor and \( \chi_{t}^{F,F} \) as the share of foreign stock market (orange tree) held by foreign investor.

**Equity constraint**  Importantly, equity constraint for home country:

\[
0 \leq \chi_{t}^{H,F} \leq \chi^{H,F}
\]

This equation is saying that home investor can not hold more than \( \chi^{H,F} \) share of foreign equity, nor short-sell foreign equity\(^7\).

And similarly we have equity constraint for foreign country:

\[
0 \leq \chi_{t}^{F,H} \leq \chi^{F,H}
\]

That is, foreign investor can not hold more than \( \chi^{F,H} \) share of home equity, nor short-sell home equity.

**Sovereign Bond**  Each country can issue (sovereign) bond in zero net supply. The bonds, denoted as \( B_{t}^{H} \) and \( B_{t}^{F} \), are instantaneously *risk-free in domestic goods* but *not* risk-free in terms of numeraire\(^8\). The price of home (foreign) bond \( B_{t}^{H} \) (\( B_{t}^{F} \)) in terms of numeraire is the same as the

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\(^6\)An increase of \( e_t \) corresponds to an appreciation of home currency relative to foreign currency.

\(^7\)Here one can replace the lower bound 0 to a negative number, say \( \chi^{F} \). The key thing is that \( \chi_{t}^{H,F} \) (the share of foreign equity held by home investor) is lower bounded.

\(^8\)Their returns are subject to exchange rate risks through price changes
price of home (foreign) good $p_t^H$ ($p_t^F$). Denote $B_t^{H,H}$ as the home bond held by home investors and $B_t^{F,H}$ as the home bond held by foreign investors. And denote $B_t^{H,F}$ as the foreign bond held by home investors and $B_t^{F,F}$ the foreign bond held by foreign investors.

**Asset returns** We introduce notations for asset returns which are *endogenous* processes. Recall that $B_t^H$ is instantaneously risk-less bond in home good and denote the return process of home bond (in terms of numeraire) as:

$$dr_t^{B_H} = \frac{d(p_t^H B_t^H)}{p_t^H B_t^H} = (\mu_{p_t^H,t} + r_t^H) dt + \sigma_{p_t^H,t} dZ_t$$

where $\mu_{p_t^H,t}$ and $\sigma_{p_t^H,t}$ are given by the endogenous process

$$\frac{dp_t^H}{p_t^H} = \mu_{p_t^H,t} dt + \sigma_{p_t^H,t} dZ_t$$

Similarly denote the return process of foreign bond (in terms of numeraire) as:

$$dr_t^{B_F} = \frac{d(p_t^F B_t^F)}{p_t^F B_t^F} = (\mu_{p_t^F,t} + r_t^F) dt + \sigma_{p_t^F,t} dZ_t$$

Recall that $S_t^H$ is the total value of home equity and define $q_t^H$ as the per unit price of home equity in terms of numeraire $Y_t$, that is, $S_t^H = q_t^H Y_t$. And postulate the endogenous process of $q_t^H$ as follows

$$\frac{dq_t^H}{q_t^H} = \mu_{q_t^H,t} dt + \sigma_{q_t^H,t} dZ$$

(5)

The return of home equity in terms of numeraire is given by

$$dr_t^{S_H} = \frac{p_t^H Y_{H,t}}{q_t^H Y_t} dt + \frac{d(q_t^H Y_t)}{q_t^H Y_t}$$

and similarly

$$dr_t^{S_F} = \frac{p_t^F Y_{F,t}}{q_t^F Y_t} dt + \frac{d(q_t^F Y_t)}{q_t^F Y_t}$$

**Forward market** Since there is no friction on the bond markets, there naturally exists a FX forward market for home bond and foreign bond. Home investor can enter a FX forward contract (long in home currency and short in foreign currency) with zero cost today which will deliver an instantaneous return $dr_t^{B_H} - dr_t^{B_F}$.

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9 $B_t^{H,H} > 0$ means lending and $B_t^{H,H} < 0$ means borrowing

10 In equilibrium, investors will exactly do so as discussed in appendix.
Wealth and portfolio shares  we introduce notations for wealth and portfolio shares, which will be determined in equilibrium. Denote the aggregate wealth of home country as \( W_t^H \) and the aggregate wealth of foreign country as \( W_t^F \).

Denote \( \theta_t^{H,S} = \frac{\chi^{H,H} S_t^H}{W_t^H} \) as the portfolio share of home equity for home country, \( \theta_t^{H,F} = \frac{\chi^{H,F} S_t^F}{W_t^H} \) as the portfolio share of foreign equity for home country. And similarly, denote \( \theta_t^{F,S} = \frac{\chi^{F,H} S_t^H}{W_t^F} \) as the portfolio share of home equity for foreign country and \( \theta_t^{F,F} = \frac{\chi^{F,F} S_t^F}{W_t^F} \) as the portfolio share of foreign equity for foreign country.

We can similarly define portfolio shares of bonds in home and foreign country. Denote \( \theta_t^{H,B} = \frac{p_H t B_t^H}{W_t^H} \) as the portfolio share of home bond for home country, \( \theta_t^{H,B} = \frac{p_H t B_t^H}{W_t^H} \) as the portfolio share of foreign bond for home country, And similarly, denote \( \theta_t^{F,B} = \frac{p_F t B_t^F}{W_t^F} \) as the portfolio share of home bond for foreign country and \( \theta_t^{F,F} = \frac{p_F t B_t^F}{W_t^F} \) as the portfolio share of foreign bond for foreign country.

Country Size  Define relative size of home country as follows.

\[
s_t = \frac{a \frac{1}{\eta} Y_{H,t}^{\frac{\eta - 1}{\eta}}}{a \frac{1}{\eta} Y_{H,t}^{\frac{\eta - 1}{\eta}} + (1 - a) \frac{1}{\eta} Y_{F,t}^{\frac{\eta - 1}{\eta}}} = \frac{1}{a} \left( \frac{Y_{H,t}}{Y_t} \right)^{\frac{\eta - 1}{\eta}} \tag{6}
\]

Optimization problems  The optimization problem for home country is as follows:

\[
\max_{\{C_{H,t}, C_{H,F,t}, \chi_{H,H,t}, \chi_{H,F}, \theta_{H,B}, \theta_{H,F} \}} \quad \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \log \left( \frac{a \frac{1}{\eta} C_{H,t}^{\frac{\eta - 1}{\eta}} + (1 - a) \frac{1}{\eta} C_{H,F,t}^{\frac{\eta - 1}{\eta}}}{\frac{1}{\eta} Y_t^{\frac{\eta - 1}{\eta}}} \right) dt \right]
\]

s.t.

\[
\frac{dW_t^H}{W_t^H} = \chi_{H,H} S_t^H dt + \chi_{H,F} S_t^F dt + \theta_{H,B} \frac{p_H t B_t^H}{W_t^H} dt + \theta_{H,F} \frac{p_F t B_t^F}{W_t^F} dt
\]

\[
1 = \frac{\chi_{H,H} S_t^H}{W_t^H} + \frac{\chi_{H,F} S_t^F}{W_t^H} + \theta_{H,B} + \theta_{H,F}
\]

\[
0 \leq \chi_{H,F} \leq \chi_{H,F} \tag{7}
\]

The optimization problem for foreign country is similar and discussed in appendix.

Market clearing conditions  Home equity market clears,

\[
\chi_{H,H} + \chi_{F,H} = 1 \tag{8}
\]
Foreign equity market clears,
\[ \chi_t^{H,F} + \chi_t^{F,F} = 1 \]  
(9)

Home bond market clears,
\[ B_t^{H,H} + B_t^{F,H} = 0 \]  
(10)

And foreign bond market clears,
\[ B_t^{H,F} + B_t^{F,F} = 0 \]  
(11)

Total consumption of home (foreign) good equals total production of home (foreign) good,
\[ C_{HH,t} + C_{FH,t} = Y_{H,t} \]  
(12)
\[ C_{HF,t} + C_{FF,t} = Y_{F,t} \]  
(13)

4 Complete market model

Before solving the model with equity constraints, it is useful to solve for the complete market case which works as a clear illustration of the country size spillover effect.

Postulate two stochastic discount factor processes for the two countries, \( \xi_{H,t} = e^{-\rho_t \frac{1}{C_{H,t}}} \) and \( \xi_{F,t} = e^{-\rho_t \frac{1}{C_{F,t}}} \), as
\[
\frac{d\xi_{H,t}}{\xi_{H,t}} = -r_{H,t} dt - m_{H,t}^{T} dZ_t \\
\frac{d\xi_{F,t}}{\xi_{F,t}} = -r_{F,t} dt - m_{F,t}^{T} dZ_t 
\]
respectively. \( m_{H,t} \) is the vector of risk prices in home country and also the consumption risk\(^{11} \) in the logarithmic utility case.

In the complete market case, there exists a unique stochastic discount factor \( \xi_t \), such that
\[
\frac{d\xi_{H,t}}{\xi_{H,t}} = \frac{d\xi_{F,t}}{\xi_{F,t}} = \frac{d\xi_{t}}{\xi_{t}} .
\]

4.1 Sovereign bond safety and country size spillover effect

**Proposition 1** (Sovereign bond safety). *Expected return difference between home bond and foreign bond is given by*
\[
\mathbb{E}_t \left[ \frac{dr_t^{BH} - dr_t^{BF}}{dt} \right] = m_{H,t}^{T} \sigma_t^e = m_{F,t}^{T} \sigma_t^e
\]
where \( dr_t^{BH} \) is the return process for home bond, \( dr_t^{BF} \) is the return process for foreign bond, \( \sigma_t^e \) is the exchange rate risk.

\(^{11}\) Consumption risk of a country is defined as the volatility vector of consumption process of that country, \( \frac{dC_{H,t}}{C_{H,t}} \).
If home country’s consumption risk is positively correlated with exchange rate risk (domestic consumption is low when domestic currency depreciates), then home bond earns a positive risk premium.

If home country’s consumption risk is negatively correlated with its exchange rate (domestic consumption is high when domestic currency depreciates), then home bond earns a negative safety premium.

Proof. see appendix

The intuition is as follows: A bond is considered safe if it has high value when consumption is low, because the bond insures investors against bad times. In our example, US treasury pays lower expected return than UK government bond if GBP depreciates against USD when consumption is low. Because in this case, US treasury is a good hedge for consumption risk and is viewed as safe, while UK government bond does not hedge consumption risk and is viewed as risky. Lustig and Verdelhan (2007) provides empirical evidence for proposition 1.

In our model, uncertainty comes from production fluctuations of the trees. When UK production declines due to a negative shock, the supply of UK good declines, and the relative price of UK good should go up, implying a higher expected return of UK bond. However, this is not the whole story for bond safety. Because the final consumption is an aggregate of both countries’ goods, another competing force emerges: the demand for US good increases because of consumption smoothing motive. The final consumption shifts more towards US good than before due to a supply drop of UK good. This positive demand shock for US good will put upward pressure on the expected return of US bond. The next proposition shows that the relative magnitude of the supply force and demand force is determined by the relative country size.

**Proposition 2** (Country size spillover effect). Solving for (14), we have

\[
\mathbb{E}_t \left[ \frac{dr^B_H - dr^B_F}{dt} \right] = \overline{\sigma}_t \sigma^e_t = \begin{bmatrix} s_t \sigma_H & (1 - s_t) \sigma_F \end{bmatrix} \begin{bmatrix} -\frac{1}{\eta} \sigma_H \\ \frac{1}{\eta} \sigma_F \end{bmatrix}
\]

\[
= \frac{1}{\eta} (-s_t \sigma_H^2 + (1 - s_t) \sigma_F^2)
\]

and the safety threshold

\[
s_C = \frac{\sigma_F^2}{\sigma_H^2 + \sigma_F^2}
\]

If \( s_t < s_C \), home country is a relatively small country and home bond is riskier than foreign bond.

If \( s_t > s_C \), home country is a relatively large country, country bond is safer than foreign bond.

Proof. see appendix
Country size spillover effect states that larger country’s bond is safer. US treasury is safer than UK government bond because the size of US economy is a larger than the size of UK economy. Since US contributes a larger share to world consumption, the world consumption risk also consists largely US risk. US treasury becomes a safe asset and pays lower expected return because it is a better hedge for world consumption risk. This is often referred to as the exorbitant privilege of the US Dollar. Going back to the supply force and demand force discussed earlier: the larger the country’s share in the world consumption, the larger the magnitude of supply or demand change of its good. When the small country becomes smaller, the large country’s production becomes more dominant in world consumption, strengthening the hedging benefit of the large country’s bond. On the other hand, when the small country grows larger, world consumption depends less on the large country’s production, reducing the hedging benefit of the large country’s bond. Country size spillover effect is stronger when there is more asymmetry in $s_t$ and $1 - s_t$. Expected return differences of sovereign bonds are sizable and persistent across countries as discussed in Hassan (2013), which also provides empirical evidence for country size spillover effect.

Figure 4: safety region of home and foreign bond

![Figure 4: safety region of home and foreign bond](image)

**Note:** This figure shows the safety region of home and foreign bond as the relative size of home country $s_t$ changes. $s^C$ is the safety threshold where UIP changes sign.
Figure 5: Expected return difference of home and foreign bond, complete market

Note: The y-axis of this figure shows the expected return difference of home and foreign bond. The x-axis is the relative size of home country $s$. The dotted vertical line $s^C$ represents the endogenous threshold for UIP changing sign. Parameter values used: growth volatility $\sigma_H^2 = \sigma_F^2 = \sigma^2 = 0.04$, and trade elasticity $\eta = 4$.

4.2 Persistence and asymmetry

In the simplest complete market case, on top of country size spillover effect, there are also persistence effect and asymmetry effect on prices of risks through changes in country size $s_t$.

The volatility $^{12}$ of country size $s_t$ and the prices of risks during normal times are given by

$$\sigma_{s_t} = \frac{\eta - 1}{\eta} (1 - s_t) \begin{bmatrix} \sigma_H \\ -\sigma_F \end{bmatrix}$$ (18)

and

$$m_{H,t} = m_{F,t} = \sigma_t = \begin{bmatrix} \frac{s_t \sigma_H}{(1 - s_t) \sigma_F} \\ \text{price of home country risk} \end{bmatrix}$$ (19)

Persistence A temporary negative shock to home country’s production immediately reduces home country’s relative country size, decreases the price of home country risk. In addition, a smaller $s_t$ also affects the magnitude of future shocks on country size $s_t$ as well as prices of risks $\sigma_t$.

$^{12}$Throughout this paper, we denote the volatility of the process $\frac{dX_t}{X_t}$ as volatility of $X_t$. 

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Unlike classic works in macro-finance literature (Bernanke et al. (1999) etc), where the persistence of a temporary shock is due to changes in current and future investment, the persistence here in our benchmark model without investment is purely from changes in country size $s_t$.

**Asymmetry**  A negative shock to home country production affects both price of home country risk and price of foreign country risk through country size spillover effect. This shock affect prices of risks asymmetrically through changes in country size $s_t$. As in equation (19), the decline of home risk price is mitigated by the smaller size of home country, $s_t$, while increase of foreign country risk price is amplified by the larger size of foreign country, $1 - s_t$. The same shock thus affects prices of the two countries’ bonds and equities asymmetrically.

5  Model with equity constraints

Adding another key ingredient to the model, the equity constraints, we proceed in two steps.

First step, we explore what happens with only one equity constraint for foreign country’s holding of home equity, $0 \leq \chi_{t}^{F,H} \leq \chi_{t}^{F,H}$. There is a maximum limit on home equity share held by foreign investors and no short-selling of home country’s equity is allowed. As shown in the following proposition 3, the two countries can still perfectly share consumption risk and have the same prices of risks as in the complete market case.

Second step, we explore the full model with equity constraints for both countries. There exists an endogenous crisis regime in the model which results in asymmetry and instability of the system.

To highlight the mechanism and simplify some algebra for illustration purpose, we assume symmetric parameters for the two trees $\mu_1 = \mu_2 = \mu$ and $\sigma_H = \sigma_F = \sigma$ hereafter. And taking advantage of symmetry, we focus on analysing home country (Home). The symmetric assumption makes sense in the example of UK and US, as the two countries have similar growth rates and volatilities. And we will focus on empirically relevant case where home country (UK) is a small country relative to foreign country (US).

5.1 Normal regime

**Proposition 3.** With only one equity constraint for foreign country’s holding of home equity, $0 \leq \chi_{t}^{F,H} \leq \chi_{t}^{F,H}$, the two countries can perfectly share consumption risk and replicate the complete market case result in the sense that proposition 1 and proposition 2 still hold true.

*Proof. see appendix*  

An intuitive way to look at proposition 3 is to count the risks and assets. There are two sources of risks, from the two trees. Even with one equity holding constraint, there are still another three assets that can be freely traded which can span all the possible states of the world. So investors in
the two countries can still replicate first best risk-sharing through portfolio re-balancing. Similar to the complete market case, there is indeterminacy in the model with respect to portfolio holdings but not asset returns.

**Proposition 4.** With the only equity constraint for foreign country’s holding of home equity, \( 0 \leq \chi_{t}^{F,H} \leq \chi^{F,H} \), a special specification for discount rate \( \rho = \left( \frac{\eta-1}{\eta} \sigma \right)^2 \), initial condition \( s_0 \), home country’s equity shares and bond holdings are given by

\[
\begin{align*}
\chi_{t}^{H,H} & \in [1 - \chi^{F,H}, 1] \\
\chi_{t}^{H,F} & = \frac{n_0 - \chi_{t}^{H,H} \rho q_{t}^{H}}{1 - \rho q_{t}^{H}} \\
q_{t}^{H,BH} & = \rho (q_{t}^{H})'(s_t) s_t (1 - s_t) (\chi_{t}^{H,H} - \chi_{t}^{H,F})(\eta - 1) \\
q_{t}^{H,BF} & = -q_{t}^{H,BH}
\end{align*}
\]

where

\[
q_{t}^{H}(s) = \frac{1}{2 \rho} \left( 1 + \frac{1 - s}{s} \ln(1 - s) - \frac{s}{1 - s} \ln(s) \right)
\]

is the per unit price of home equity and taking derivatives with respect to \( s_t \), we have

\[
(q_{t}^{H})'(s_t) = -\frac{1}{2 \rho} \frac{1}{s(1 - s)} \left( 1 + \frac{1 - s}{s} \ln(1 - s) + \frac{s}{1 - s} \ln(s) \right)
\]

And \( n_0 = q^{H}(s_0) \) is the initial wealth share of home country.

**Proof.** see appendix
Figure 6: Equity shares and portfolio shares of home investors

![Graphs showing equity and portfolio shares of home investors.]

Note: This panel of figures present the equity shares and portfolio shares of home investors solved from the model. For all panels, the x-axis is the relative size of home country s. For the top left panel, the equity share $\chi_{H,H}$ is defined as home investors’ holdings of home equities over the total supply of home equities; and the portfolio share of home equity $\theta_{H,S}^{H}$ is defined as home investors’ holdings of home equities over their total wealth. For top right panel, $\chi_{H,F}$ is home investors’ holdings of foreign equities over the total supply of foreign equities; and the portfolio share of foreign equity $\theta_{H,S}^{F}$ is defined as home investors’ holdings of foreign equities over their total wealth. For the bottom left panel, $\theta_{H,B}^{H}$ is defined as home investors’ holdings of home bond over their total wealth And for the bottom right panel, $\theta_{H,B}^{F}$ is defined as home investors’ holdings of foreign bond over their total wealth.

Parameter values used: volatility of GDP growth $\sigma^2 = 0.04$, trade elasticity $\eta = 2$, and equity share $\chi_{H,H} = 1$.

From proposition 4 and figure 6, we see that the net borrowing in bonds between the two countries is zero. The two countries smooth their consumption by holding equities and use bonds to help achieve perfect risk-sharing. Both countries go short in foreign bond and long in domestic bond. Because domestic bond is a better hedge for domestic risk, the two countries can offload the extra domestic risk from domestic equity holding requirement by lending in domestic bond and borrowing in foreign bond.

Comparing figure 5 and the right-bottom panel of figure 6, we see that countries borrow more in foreign bond when their country size grows larger and domestic bond becomes safer, fixing home
country’s holding of domestic equity share. Because when a country grows larger, its domestic equity price increases, leading to a heavier portfolio weight on domestic equity and thus more domestic risk exposure, which requires more hedging. This is consistent with the empirical fact documented by Du, Pflueger, and Schreger (2020). Until now, the model with only country size spillover effect can explain the UIP violation in normal times and find empirical support from earlier work. However, the model does not have space for crisis yet and is thus silent about what happens in crisis.

5.2 Crisis regime

Moving on to second step, with equity holding constraints for both countries, an endogenous crisis regime emerges and the system moves into the crisis regime when one country falls too small.

Proposition 5 (Crisis regime). The system moves in to crisis regime if \( s_t \in [0, s^U] \cup [1 - s^U, 1] \), where \( s^U \) is the crisis boundary and solves

\[
q_1(s^U) = \frac{n_0 - \bar{X}_{H,F}}{\rho(1 - \bar{X}_{H,F})} \tag{20}
\]

If \( s_t \in [0, s^U] \), we have

\[
\chi_{t}^{H,H} = 1, \quad \chi_{t}^{H,F} = \bar{X}_{H,F}
\]

If \( s_t \in [1 - s^U, 1] \), we have

\[
\chi_{t}^{H,H} = 1 - \bar{X}_{F,H}, \quad \chi_{t}^{H,F} = 0
\]

Proof. see appendix.

The crisis boundary \( s^U \) is the left margin where both countries’ equity holding constraints bind and \( 1 - s^U \) is the right margin where the equity holding constraints bind in the opposite direction. When \( s_t < s^U \), the two countries can perfectly share exchange rate risk through freely adjusting their equity holdings and trading on the FX market. The gains and losses from FX market will be delivered by capital flows induced by equity trading, until both equity constraints bind. we refer to the constrained region as crisis regime. In crisis regime, risk-sharing is limited, and asset returns vary discontinuously from in normal regime due to the constraint on equity rebalancing.

5.2.1 Safety spectrum

In crisis regime, equity reblancing is constrained and risk-sharing is limited. This market segmentation drives a wedge between normal time SDF and crisis time SDF, thus a wedge of risk prices
between normal times and crisis time. We refer to the effect of the constraints on equity rebalancing as the equity rebalancing effect\textsuperscript{13}.

**Proposition 6** (equity rebalancing effect). *In crisis region, there exists a wedge between the normal time SDF and crisis time SDF, due to lack of equity rebalancing. For home country investors, denote this wedge as $\sigma_{nt}$. If $0 < s_t < s^U$,\footnote{To be precise, this is “the lack of equity rebalancing” effect}*

$$\sigma_{nt} = \frac{(1 - \chi_{H,F})s_t - \eta - 1}{(1 - \chi_{H,F})s_t + \chi_{H,F}} \eta (1 - s_t) \begin{bmatrix} \sigma_H \\ -\sigma_F \end{bmatrix}$$

(21)

If $1 - s^U < s_t < 1$,

$$\sigma_{nt} = \frac{\eta - 1}{\eta} (1 - s_t) \begin{bmatrix} \sigma_H \\ -\sigma_F \end{bmatrix}$$

(22)

symmetrically for foreign investors.

*Proof. see appendix*

With two equity constraints and the crisis region, the model exhibits a safety spectrum for each country with four regions identified by three key safety thresholds.

**Proposition 7** (Safety spectrum). *With equity holding constraints for both countries, $0 \leq \chi_{F,H} \leq \bar{\chi}_{F,H}$ and $0 \leq \chi_{H,F} \leq \bar{\chi}_{H,F}$, and reasonable parameter restrictions on $(\bar{\chi}_{F,H}, \bar{\chi}_{H,F}, \eta)$, there are three key thresholds, normal time safety threshold $s_C$, crisis boundary $s^U$, and crisis time safety threshold $s^A$,

$$s_C = \frac{1}{2}$$

(23)

$$q^H(s^U) = \frac{n_0 - \bar{\chi}_{H,F}}{\rho(1 - \bar{\chi}_{H,F})}$$

(24)

$$\frac{2(1 - \bar{\chi}_{H,F})(s^A)^2 + (2\bar{\chi}_{H,F} + (1 - \bar{\chi}_{H,F})(\eta - 2))s^A - \eta\bar{\chi}_{H,F}}{(1 - \bar{\chi}_{H,F})s^A + \bar{\chi}_{H,F}} = 0$$

(25)

such that

$$0 < s^A < s^U < s_C$$

(26)

When $s^U < s_t < 1 - s^U$, the system stays in normal regime, country size determines bond safety as in proposition 2:

If $s^U < s_t < s_C$, home country is a relatively small country, home country’s bond is risky.

If $s^C < s_t < 1 - s^U$, home country is a relatively large country, home country’s bond is safe.

When $0 < s_t < s^U$ or $1 - s^U < s_t < 1$, the system moves into crisis regime. Country size spillover effect and the equity rebalancing effect will jointly determine sovereign bond safety.
When $0 < s_t < s^U$, the system moves into crisis regime where country size spillover effect competes with equity rebalancing effect:

If $s^A < s_t < s^U$, equity rebalancing effect dominates, home country’s bond is safe for domestic investors.

If $0 < s_t < s^A$, country size spillover effect dominates, home country’s bond is risky for domestic investors.

When $1 - s^U < s_t < 1$, the system moves into crisis regime where country size spillover effect joins forces with equity rebalancing effect: home country’s bond is safe for domestic investors.

Proof. see appendix

Figure 7: safety region of home and foreign bond for home investors

Note: This figure shows the safety region of home and foreign bond as the relative size of home country $s_t$ changes, from the perspective of home investors. $s^U$ is the boundary of crisis regime (constrained) as home country falls in relative country size, symmetrically for $1 - s^U$. $s^C$ and $s^A$ are the safety thresholds where UIP changes sign in normal regime (unconstrained) and crisis regime (constrained).
Figure 8: Expected return difference of home and foreign bond for home investors

Note: The $y$-axis of this figure shows the expected return difference of home and foreign bond from the perspective of home investors. The $x$-axis is the relative size of home country $s$. The dotted blue line (unconstrained) corresponds to the complete market model. The solid orange line (constrained) corresponds to the full model with two equity constraints. The three dotted vertical lines (from left to right) $s^A$, $s^U$, $s^C$ represent the endogenous threshold for UIP changing sign in crisis regime, the boundary of crisis regime, and the threshold for UIP changing sign in normal regime, respectively. Parameter values used: $\eta = 4$, $\chi^{F,H} = \chi^{H,F} = 0.2$, $\sigma^2 = 0.04$.

As shown in figure 7, cut by the three thresholds, there are four regions along the safety spectrum. Blue regions represent where foreign bond is safer for domestic investors, and green regions represent where domestic bond is safer. Figure 8 shows the expected return difference between domestic bond and foreign bond for home country investors.

In normal regime where $s^U < s_t < 1 - s^U$, there is only the familiar country size spillover effect: the larger country’s bond is safer. If home country’s size continues falling below $s^U$, the risk-sharing is limited by the equity holding constraints.

In crisis regime, investors in both countries are forced to hold more domestic risk and less foreign risk compared to the perfect risk-sharing scenario in normal regime. Because of limited risk-sharing, domestic bond becomes safer for domestic investors in crisis regime than in normal regime, as it is a better hedge for domestic risk.

Equity rebalancing effect improves safety of the domestic bond for domestic investors in crisis regime while country size spillover effect improves safety of the larger country’s bond. So in crisis regime, country size spillover effect competes with equity rebalancing effect for the smaller country’s investors but collaborates for the larger country’s investors.

If the smaller country’s size falls in between $s^A$ and $s^U$, equity rebalancing effect dominates.
country size spillover effect. The smaller country’s domestic bond is safe for domestic investors. If the smaller country falls below \( s^A \), country size spillover effect dominates equity rebalancing effect. The smaller country’s domestic bond is risky for domestic investors.

Whereas for the larger country when \( 1 - s_t > s^C \), its domestic bond is always safe for domestic investors. The safety of the larger country’s domestic bond is discontinuously strengthened when \( 1 - s_t > s^U \) due to equity rebalancing effect.

5.2.2 Domestic amplification

Equity rebalancing amplifies the effect of domestic shock. This domestic amplification exists both in the “time series” (compared to normal regime) and in the “cross section” (compared to foreign country).

In the crisis regime \([0, s^U]\), the prices of risks for home country investors is given by

\[
m_{H,t} = \frac{(1 - \bar{\chi}^{H,F})s_t}{(1 - \bar{\chi}^{H,F})s_t + \bar{\chi}^{H,F} \sigma_s} + \sigma_t = \begin{cases} \frac{(1 - \bar{\chi}^{H,F})}{(1 - \bar{\chi}^{H,F})s_t + \bar{\chi}^{H,F} \sigma_s} \left[ \frac{\eta - 1}{\eta} (1 - s_t) + 1 \right] s_t \sigma_H \\ \geq s_t \sigma_H, \text{ normal time price of home country risk} \\ \frac{1}{1 - \bar{\chi}^{H,F}} \left[ \frac{\eta - 1}{\eta} \eta \right] (1 - s_t) \sigma_F \\ \leq (1 - s_t) \sigma_F, \text{ normal time price of foreign country risk} \end{cases} \tag{27} \]

and the prices of risks for foreign country investors is given by

\[
m_{F,t} = \sigma_{1 - s_t} + \bar{\sigma}_t = \begin{cases} \frac{1}{\eta} s_t \sigma_H \\ \leq s_t \sigma_H, \text{ normal time price of home country risk} \\ \frac{\eta - 1}{\eta} s_t + (1 - s_t) \sigma_F \\ \geq (1 - s_t) \sigma_F, \text{ normal time price of foreign country risk} \end{cases} \tag{28} \]

For home country investors, compared to in normal times, the effect of a shock on domestic risk price is amplified by the factor

\[
\frac{(1 - \bar{\chi}^{H,F})}{(1 - \bar{\chi}^{H,F})s_t + \bar{\chi}^{H,F} \sigma_s} \frac{\eta - 1}{\eta} (1 - s_t) + 1 > 1
\]

and the effect of a shock on foreign risk price is mitigated by the factor

\[
1 - \frac{(1 - \bar{\chi}^{H,F})s_t}{(1 - \bar{\chi}^{H,F})s_t + \bar{\chi}^{H,F} \sigma_s} \frac{\eta - 1}{\eta} < 1
\]

Similarly for foreign country investors, the effect of a shock on domestic risk price is amplified by
the factor
\[
\frac{\eta - 1}{\eta} \frac{s_t}{1 - s_t} + 1 > 1
\]
and the effect of a shock on foreign risk price is mitigated by the factor
\[
\frac{1}{\eta} < 1
\]
In crisis regime, domestic risk price response to a shock is amplified and foreign risk price response to a shock is mitigated compared to in normal times due to lack of equity rebalancing. Domestic amplification improves the hedging benefit of domestic bond in crisis compared to in normal times.

In another dimension, comparing the prices of risks between home country investors and foreign country investors, we have
\[
\left(1 - \frac{1}{\chi_{H,F}}\right) \left(1 - \frac{1}{\chi_{H,F}}\right) s_t \sigma_H > s_t \sigma_H > \frac{1}{\eta} s_t \sigma_H
\]
where the first term is domestic risk price for home country investors in crisis regime \([0, s^U]\), the second term is home country risk price in normal regime \([s^U, s^C]\), and the third term is foreign risk price for foreign country investors in crisis regime \([0, s^U]\). And similarly
\[
\left[1 - \frac{(1 - \chi_{H,F})s_t}{(1 - \chi_{H,F})s_t + \chi_{H,F}} \frac{1}{\eta} \right] (1 - s_t) \sigma_F < (1 - s_t) \sigma_F < \left[\frac{\eta - 1}{\eta} s_t + (1 - s_t)\right] \sigma_F
\]
where the first term is foreign risk price for home country investors in crisis regime \([0, s^U]\), the second term is foreign country risk price in normal regime \([s^U, s^C]\), and the third term is domestic risk price for foreign country investors in crisis regime \([0, s^U]\). In crisis regime, domestic investors hold more domestic risk than foreign investors and thus require a higher risk premium than foreign investors. Domestic amplification drives up domestic risk price and pushes down foreign risk price in crisis for investors in both countries.

5.2.3 Domestic and global safety

In crisis regime, because of domestic amplification, domestic safety status of bonds may or may not coincide with global safety status of bonds. As shown in the following proposition 8, the smaller country’s bond is domestically safe in mild crisis when country sizes are mildly asymmetric and the larger country’s bond is globally safe in deep crisis when country sizes are sufficiently asymmetric.

**Proposition 8** (Domestic and global safety). Assume that home country is the smaller country, \(s_t < s^C\).

The smaller country’s bond is domestically safe if \(s_t \in [s^A, s^U]\).

The larger country’s bond is globally safe if and only if \(s_t \in [0, s^A] \cup [s^U, s^C]\).
Proof. see appendix.

Figure 9: Global safety region of home and foreign bond

Note: This figure shows the safety region of home and foreign bond as the relative size of home country $s_t$ changes, from the perspective of both home (the upper axis) and foreign investors (the lower axis). $s^U$ is the boundary of crisis regime (constrained) as home country falls in relative country size, symmetrically for $1 - s^U$. $s^C$ and $s^A$ (symmetrically for $1 - s^A$) are the safety thresholds where UIP changes sign in normal regime (unconstrained) and crisis regime (constrained).
Figure 10: Expected return difference of home and foreign bond for home and foreign investors

Note: The y-axis of this figure shows the expected return difference of home and foreign bond. The x-axis is the relative size of home country $s$. The solid blue line (constrained, home investor) and the dotted orange line (constrained, foreign investors) shows the expected return difference of home and foreign bond in the full model (with two equity constraints) from the perspective of home investors and foreign investors, respectively. The dotted yellow line (unconstrained) shows the expected return difference of home and foreign bond in complete market model. And the three dotted vertical lines (from left to right) $s^A = 0.225$, $s^U = 0.34$, $s^C = 0.5$ represents the endogenous threshold for UIP changing sign in crisis regime, the boundary of crisis regime, and the threshold for UIP changing sign in normal regime, respectively. Parameter values used: $\eta = 4$, $\chi^{F,H} = \chi^{H,F} = 0.2$, $\sigma^2 = 0.04$.

In crisis regime, biding equity constraints drives a wedge between SDFs of home and foreign investors. Different pricing kernels result in different returns for the same asset. Investors in the two countries disagree on expected return difference between bonds in crisis, as shown in figure 10. Technically, the heterogeneity in SDFs comes from heterogeneity in constraints. The two countries face asymmetric complementary margin requirements on their equity holdings which results in asymmetric Lagrangian multipliers associated with the binding constraints.

For the smaller country, when falling into crisis, country size spillover effect competes with equity rebalancing effect. In mild crisis, equity rebalancing effect dominates country size spillover effect. The smaller country’s bond becomes domestically safe. In deep crisis, country size spillover effect dominates equity rebalancing effect. The larger country’s bond becomes safe for investors in the smaller country. For investors in the larger country, domestic bond is always safe and domestic bond safety status gets strengthened upon entering crisis regime, see the jumps in figure 10. So the larger country’s bond is globally safe in normal times and in deep crisis, as shown in figure 9.
5.2.4 Market segmentation

**Proposition 9** (Market segmentation). In crisis regime, bond holdings of the two countries are given by

\[
\theta_t^{H,B} = \theta_t^{H,B} = 0, \quad \theta_t^{F,B} = \theta_t^{F,B} = 0
\]

**Proof.** see appendix

Forced by the constraints, investors in both countries hold more domestic risk than desired and would like to offload domestic risk to foreign investors. However, no such security is available because domestic amplification is resulted from dispersion in consumption prices (in terms of numeraire) of the two countries created by binding equity constraints and applies to any real asset. Real bond returns for investors bear the extra domestic risk coming from consumption price and no bond is held or traded between the two countries in crisis even though there is no friction in bond markets. Liquidity drains between the two countries in every asset market\(^{14}\) and financial dichotoour emerges in crisis regime.

5.2.5 Non-linearity and systemic risk

**Non-linearity** The non-linearity in asset returns shows up for investors in the smaller country who hold both domestic equity and foreign equity. The non-linearity factor for home country is given by

\[
\frac{(1 - \overline{X}^{H,F})s_t}{(1 - \overline{X}^{H,F})s_t + \overline{X}^{H,F}}
\]

Taking derivative with respect to \(s_t\), we have

\[
\overline{X}^{H,F}(1 - \overline{X}^{H,F})
\]

\[
\left[(1 - \overline{X}^{H,F})s_t + \overline{X}^{H,F}\right]^2
\]

where we see the non-linearity effect gets stronger when \(s_t\) is smaller, consistent with figure 8.

**Systemic risk** At crisis boundaries \(s^U\) and \(1 - s^U\), there are endogenous jumps between normal regime and crisis regime for both countries, which is the systemic risk in the model. The discontinuous change in expected return difference between sovereign bonds for home country investors at \(s^U\) (in absolute value), denoted as \(\Delta_{11}\) is given by

\[
\Delta_{11} = \frac{(1 - \overline{X}^{H,F})s^U}{(1 - \overline{X}^{H,F})s^U + \overline{X}^{H,F}} \frac{\eta - 1}{\eta^2} (1 - s^U)(\sigma_H^2 + \sigma_F^2)
\]

\(^{14}\)There is still trade happening between the two countries and potential trade in assets within domestic investors.
and similarly denote $\Delta_{21}$ as the absolute change in expected return difference between sovereign bonds for foreign country investors at $s^U$, where

$$\Delta_{21} = \frac{\eta - 1}{\eta^2} s^U (\sigma_H^2 + \sigma_F^2)$$ (31)

Since $s^U < \frac{1 - \chi_{H,F}}{2(1 - \chi_{H,F})}$, we have

$$\Delta_{11} > \Delta_{21}$$

that is, the smaller country suffers greater systemic risk instability when it falls into crisis.

6 UIP reversal, flight-to-safety, CIP deviations and convenience yields

The model with crisis regime can explain several puzzles strongly associated with crisis periods: UIP reversal, flight to safety, CIP deviations and convenience yields.

First, in crisis regime, both countries price risks differently from normal regime. Binding equity constraints distort asset returns and asset safety in crisis, which explains the UIP reversal and flight to safety in crisis. Second, in crisis regime, the two countries disagree on prices of risks with each other, which explains the CIP deviations and convenience yields: in crisis, UK investors and US investors perceive different returns for exactly the same bond, the US Treasury. The gap between the actual return of US Treasury and the perceived return of US Treasury by foreign investors is also known as the CIP deviations or the convenience yield of the US Treasury, which is a signature of the 2008 Great Financial Crisis (GFC).

6.1 UIP reversal

UIP reversal is the opposite direction of UIP violation in normal times. As documented in Corsetti and Marin (2020), in normal times, US Treasury is a safer asset than UK government bond for UK investors and pays lower expected return as its safety premium. While when crisis hit, UK government bond pays lower expected return and becomes safer than US Treasury for UK investors.

Mapping into the model, in normal times, UK country size $s_t$ is above the crisis boundary $s^U$ but below the normal time safety threshold $s^C$. US Treasury is safer than UK government bond because of country size spillover effect. If UK economy or US econoour suffers a rare loss, $s_t$ falls into $[s^A, s^U]$ or rises to above $s^C$, UK government bond reverses to become a domestic safe bond and pays lower expected return for domestic investors compared to the US Treasury. In addition, if $s_t$ rises to $[s^C, 1 - s^U]$ or $[1 - s^A, 1]$, UK government bond becomes a global safe bond. As shown in figure 11, UIP reversal between UK and US government bond happens when UK country size $s_t$ falls into the left blue area $[0, s^A]$ from the green area $[s^A, s^U]$. 

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The model can also speak to emerging economies. For emerging economies whose initial country size is too small and is below $s^A$, if $s_t$ rises into $[s^A, s^U]$ because the emerging econour grows larger due to its higher growth rate or the larger country, say US, falls into crisis due to rare losses, the emerging econour’s domestic bond reverses from being the riskier bond to become the safer bond than US Treasury for domestic investors. As shown in figure 11, UIP reversal happens for emerging countries when their domestic country size $s_t$ rises into the green area $[s^A, s^U]$ from the left blue area $[0, s^A]$.

![Figure 11: UIP reversal](image)

**Note:** This figure shows how UIP reversal happens as the relative size of home country $s_t$ changes. $s^A$, $s^U$, $s^C$ represents the endogenous threshold for UIP changing sign in crisis regime, the boundary of crisis regime, and the threshold for UIP changing sign in normal regime, respectively.

### 6.2 Flight to safety

In a flight-to-safety phenomena, investors demand safe assets and push down the return of safe assets in crisis. US Treasury and German bond are good examples for being the global safe asset in crisis and pay lower expected returns than in normal times.

Mapping into the model, if UK country size $s_t$ falls below $s^U$, the safety of US Treasury for US domestic investors is strengthened by equity rebalancing effect and US Treasury yield drops. If UK country size $s_t$ falls below $s^A$, US Treasury is the safe asset for both UK investors and US investors, the global safe asset. The smaller UK country size $s_t$ falls, the safer the US Treasury. Because country size spillover effect, which improves US Treasury safety, gets stronger with falling $s_t$. When UK country size $s_t$ falls into crisis regime $[0, s^U]$, US investors find US Treasury even safer than in normal times and UK investors find US Treasury the safer bond than UK government bond if $s_t$ falls below $s^A$.

In the case of the European debt crisis, German bond is the safe asset and pays historically low return. In the model, when periphery countries falls deep in crisis and their country size $s_t$ drops below $s^A$, German bond is the global safe asset and investors are willing to accept an extra low return as the safety premium.
In a deep crisis, country size spillover effect dominates equity rebalancing effect and the larger country’s bond is the global safe asset. From the point of view of investors in the smaller country, a flight-to-safety to the larger country’s bond happens when $s_t$ falls into the left blue area as shown in figure 12.

For emerging countries whose initial country size $s_t$ is in the green area $[s^A, s^U]$, if US econoour suffers from rare losses and $s_t$ rises into blue area $[s^U, s^C]$, US Treasury becomes a global safe asset. As shown in figure 12, a flight-to-safety to the larger country’s bond also happens when $s_t$ rises into the blue area $[s^U, s^C]$ from the green area $[s^A, s^U]$.

![Figure 12: Flight to safety](image)

**Note:** This figure shows how flight-to-safety happens as the relative size of home country $s_t$ changes. $s^A$, $s^U$, $s^C$ represents the endogenous threshold for UIP changing sign in crisis regime, the boundary of crisis regime, and the threshold for UIP changing sign in normal regime, respectively.

### 6.3 CIP deviations and convenience yields

CIP deviations is a failure of the Law of One Price: assets with the same underlying dividend flow pay different returns. Government bond convenience yield is the return difference between risk-free rate and government bond yield. The relative convenience yield between sovereign bonds are often related to CIP deviations. Du, Im, and Schreger (2018a) studies the US Treasury premium which is defined as the relative convenience yield between US Treasury and other countries’ government bonds by measuring CIP deviations between government bond yields. In our model, the larger country’s bond enjoys a positive convenience yield relative to the smaller country’s bond in crisis regime.

In crisis region, if home (G-10) investors want to borrow US dollar, they can not directly borrow from US dollar cash market with rate $d_{t}^{F,B^{F}}$ because of market segmentation. However, they can borrow domestic currency at rate $d_{t}^{H,B^{H}}$ and simultaneously enter a forward contract $-d_{t}^{H,B^{H}} + d_{t}^{H,B^{F}}$ to sell domestic currency for US dollar in the future. The implied US dollar rate from FX swap market (or the synthetic dollar rate) is thus $d_{t}^{H,B^{F}}$.

**Proposition 10.** The CIP condition is violated in crisis regime. The direct US dollar rate from
Cash market is lower than the synthetic dollar rate implied from FX swap market, that is

\[
E_t \left( \frac{dr_{i,t}^{F,B}}{dt} - \frac{dr_{i,t}^{H,B}}{dt} \right) = r_{F,t} - r_{H,t} - \frac{1}{1 - n_t} \sigma_n \sigma_e < 0
\]  

(32)

Proof. see appendix

7 Financial development and trade elasticity

With a stable country size \( s_t \), bond safety can also change with a shift of the safety spectrum due to changes in financial fiction parameters \( 1 - \overline{\chi}_{F,H} \), \( \chi_{H,F} \), and trade elasticity \( \eta \).

7.1 Financial development

As discussed in section 5.2, it is the foreign country’s financial development that matters for bond safety when domestic country size shrinks and falls into crisis regime. In crisis regime \([0, s^U]\), a tightening of the larger country’s equity holding constraint (i.e., a smaller \( \overline{\chi}_{H,F} \)), has impact on two safety thresholds, the crisis boundary \( s^U \) and crisis time safety threshold \( s^A \), the non-linear domestic amplification for home country investors, and the systemic risk instability. Tighter equity constraint makes it harder for consumption smoothing and risk-sharing when there is asymmetry in country sizes, which shifts \( s^U \) to the right and expands the crisis regime. Chances of entering and the time spent in the crisis regime are increased. Meanwhile, a tighter constraint strengthens equity rebalancing effect and shifts crisis time safety threshold \( s^A \) to the left. The safety region of domestic bond in crisis regime is expanded due to increased hedging benefit of domestic bond.

On the other hand, financial development (i.e., a larger \( \overline{\chi}_{H,F} \)) reduces the crisis regime coverage, as well as the safety region of domestic bond in crisis regime until the left threshold \( s^A \) exceeds the right threshold \( s^U \). With a sufficient loose constraint, equity rebalancing effect is too weak to reverse country size spillover effect and foreign bond is still the safer bond in crisis regime for domestic investors. See equation (20), (25), (21) and figure 13.
Figure 13: Changes of safety thresholds with respect to equity limit $\chi^{H,F}$

Note: This figure shows how the safety thresholds $s^C$ (UIP changing sign in normal regime, the blue line), $s^U$ (boundary of crisis region, the orange line), and $s^A$ (UIP changing sign in crisis regime, the yellow line) change with respect to equity limit $\chi^{H,F}$. The $y$-axis of this figure is the relative size of home country $s$. The $x$-axis is the tightness of equity constraint $\chi^{H,F}$ (the upper limit of foreign equity share held by home investors). The equity constraint is tighter with smaller $\chi^{H,F}$. Parameter values used: $\eta = 4$, $\chi^{F,H} = 0.2$, $\sigma^2 = 0.04$.

Financial development also matters for the non-linear domestic amplification effect and systemic risk instability, see figure 14. The larger country’s financial development reduces systemic risk instability for both countries when the smaller country falls too small, see equation (30) and (31). And the domestic non-linear effect weakens with foreign financial development, see equation (29).
Figure 14: Changes of expected return difference of home and foreign bond with respect to equity limit $\chi^{H,F}$

Note: The $y$-axis of this figure shows the expected return difference of home and foreign bond from the perspective of home investors. The $x$-axis is the relative size of home country $s$. The dotted blue line (unconstrained) corresponds to the complete market model. The solid lines corresponds to the full model with different tightness of equity constraint $\chi^{H,F}$ (the upper limit of foreign equity share held by home investors). The equity constraint is tighter with smaller $\chi^{H,F}$. Parameter values used: $\eta = 4$, $\chi^{F,H} = 0.2$, $\sigma^2 = 0.04$.

7.2 trade elasticity

A larger trade elasticity $\eta$, which means domestic good and foreign good are more substitutable, does not affect the crisis boundary $s^U$ but shifts the crisis time safety threshold $s^A$ to the left, because substitutability between two goods produced by the two countries weakens country size spillover effect and strengthens equity rebalancing effect. See equation (16), (21), and figure 15. Notice that there is discontinuity at $\chi^{H,F} = 0$ when equation (25) degenerates.
Figure 15: Changes of safety threshold with respect to trade elasticity $\eta$

Note: The $y$-axis of this figure shows the relative size of home country $s$. The $x$-axis is the tightness of equity constraint $\chi^{H,F}$ (the upper limit of foreign equity share held by home investors). ... to be completed

8 Conclusion

This paper provides a theory of sovereign bond safety which is jointly determined by country size and equity rebalancing. Country size spillover effect improves the safety of the larger country’s bond, which explains normal time UIP violation. Equity rebalancing, the equity holding constraints in the model, creates endogenous systemic risk instability between normal regime and crisis regime where domestic risk is amplified. The interaction between country size and equity rebalancing in crisis regime explains UIP reversal, flight to safety, sovereign bond CIP deviations and convenience yields at the same time.
References


Appendix

A Additional Proofs and Derivations

Trade market: Notice that international trade is a static problem for both countries. Since there is no friction in the international trade market and homogeneous preferences, we have that

\[
\frac{C_{H,H,t}}{C_{H,F,t}} = \frac{C_{F,H,t}}{C_{F,F,t}} = \frac{Y_{H,t}}{Y_{F,t}}
\]

Using market clearing condition for home good and foreign good,

\[
C_{H,H,t} + C_{F,H,t} = Y_{H,t}
\]

\[
C_{H,F,t} + C_{F,F,t} = Y_{F,t}
\]

we have

\[
C_{H,t} + C_{F,t} = Y_t
\]

where

\[
C_{H,t} = \left[ a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\eta-1}
\]

\[
C_{F,t} = \left[ a^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} \right]^{\eta-1}
\]

\[
Y_t = \left[ a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\eta-1}
\]

As a result, the prices of the two goods produced by the two trees are given by

\[
p^H_t = \left( \frac{Y_{H,t}}{a Y_t} \right)^{1/\eta} \quad \text{and} \quad p^F_t = \left( \frac{Y_{F,t}}{(1-a) Y_t} \right)^{1/\eta}
\]

We have that

\[
C_{H,t} = p^H_t C_{H,H,t} + p^F_t C_{H,F,t}
\]

\[
C_{F,t} = p^H_t C_{F,H,t} + p^F_t C_{F,F,t}
\]

Recall that country size if defined as follows

\[
s_t = \frac{a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}}}{a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}} = a^{\frac{1}{\eta}} \left( \frac{Y_{H,t}}{Y_t} \right)^{\frac{\eta-1}{\eta}}
\]

as home country’s share of the world total output, i.e. the country size of home country, which will turn out to be an important state variable. We have that

\[
p^H_t Y_{H,t} = s_t Y_t \quad \text{and} \quad p^F_t Y_{F,t} = (1-s_t) Y_t
\]

The aggregate wealth of home country is

\[
W^H_t = \chi^H_t s^H_t + \chi^F_t s^F_t + p^H_t B^H_t + p^F_t B^F_t
\]

\[
(33)
\]
The aggregate wealth of foreign country is
\[ W_t^F = \chi_t^{F,H} s_t^H + \chi_t^{F,F} s_t^F + p_t^H B_t^{F,H} + p_t^F B_t^{F,F} \]

The optimization problem for home country is as follows:

\[
\begin{align*}
\max \quad & \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \log \left( \left[ a^{\frac{n-1}{n}} C_{HH,t}^\frac{n-1}{n} + (1-a)^{\frac{1}{n}} C_{HF,t}^\frac{n-1}{n} \right]^{\frac{n}{n-1}} \right) \right] \\
\text{s.t.} \quad & \frac{dW_t^H}{W_t^H} = \frac{\chi_t^{H,H} s_t^H}{W_t^H} dt + \frac{\chi_t^{H,F} s_t^F}{W_t^H} dt + p_t^H B_t^{H,H} dt + p_t^F B_t^{H,F} dt \\
& 1 = \frac{\chi_t^{H,H} s_t^H}{W_t^H} + \frac{\chi_t^{H,F} s_t^F}{W_t^H} + \theta_t^{H,H} + \theta_t^{H,F} \\
& 0 \leq \chi_t^{H,F} \leq \chi_t^{H,F} \quad (35)
\end{align*}
\]

Define home country’s wealth share as
\[ n_t = \frac{W_t^H}{W_t^H + W_t^F} \quad (36) \]

**Lemma 1.** Home country’s wealth share \( n_t = \frac{W_t^H}{W_t^H + W_t^F} \) is a function of country size \( s_t \). And we always have that
\[ p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = 0 \quad (37) \]

**Proof.** From optimization problems, we have that
\[ C_{H,t} = \rho W_t^H \quad \text{and} \quad C_{F,t} = \rho W_t^H \]

In the aggregate, we have that total consumption equals total output
\[
\begin{align*}
C_{HH,t} + C_{HF,t} &= Y_{H,t} \\
C_{HF,t} + C_{FF,t} &= Y_{F,t}
\end{align*}
\]

And using the result from trade market optimization, we have
\[ C_{H,t} + C_{F,t} = \bar{Y}_t \]

That is,
\[ \rho W_t^H + \rho W_t^F = \bar{Y}_t \]

The total wealth in the world is given by
\[ W_t^H + W_t^F = \frac{\bar{Y}_t}{\rho} \]

There are two state variables, wealth share \( n_t \) and country size \( s_t \). In equilibrium, all the prices and quantities must be functions of state variables \( n_t \) and \( s_t \). We can rewrite home country’s wealth.
Equation (38) is an implicit function and we can solve for \( n_t \) and similarly that
\[
p_t^H Y_{H,t} = s_t \bar{Y}_t \quad \text{and} \quad p_t^F Y_{F,t} = (1 - s_t) \bar{Y}_t
\]
we have
\[
p_t^H = a \frac{1}{\eta} s_t \frac{1}{s_t - 1} \quad \text{and} \quad p_t^F = (1 - a) \frac{1}{\eta} (1 - s_t)^{-\frac{1}{\eta}}
\]
And in equilibrium \( \theta_t^{H,B} = \frac{p_t^H B_t^{H,H}}{W_t^H}, \theta_t^{H,F} = \frac{p_t^F B_t^{H,F}}{W_t^F}, q_t^H, q_t^F, \chi_t^{H,H} \) and \( \chi_t^{H,F} \) must be functions of state variables \( n_t \) and \( s_t \). We have that
\[
n_t = \frac{\chi_t^{H,H} q_t^H + \chi_t^{H,F} q_t^F}{1 - (\phi_t^{H,B} + \phi_t^{H,F})} \equiv f(s_t, n_t)
\]
Equation (38) is an implicit function and we can solve for \( n_t \) as a function of \( s_t \).

Now we have that in equilibrium \( \theta_t^{H,B} = \frac{p_t^H B_t^{H,H}}{W_t^H}, \theta_t^{H,F} = \frac{p_t^F B_t^{H,F}}{W_t^F}, q_t^H, q_t^F, \chi_t^{H,H} \) and \( \chi_t^{H,F} \) must be functions of the only state variable \( s_t \).

Recall the dynamic budget constraint of home country,
\[
\frac{dW_t^H}{W_t^H} = \frac{\chi_t^{H,H} S_t^H}{W_t^H} \frac{dY_t}{Y_t} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} \frac{dY_t}{Y_t} = \mu_t^S dt + \left( \frac{(q_t^H)(s_t)}{q_t^H} \sigma_{s_t} + \bar{\sigma}_t \right) dZ_t
\]
And the asset return processes,
\[
dr_t^{S_H} = \frac{p_t^H Y_{H,t}}{q_t^H Y_t} dt + \frac{d(q_t^H Y_t)}{q_t^H Y_t} = \mu_t^S dt + \left( \frac{(q_t^H)(s_t)}{q_t^H} \sigma_{s_t} + \bar{\sigma}_t \right) dZ_t
\]
and similarly
\[
dr_t^{S_F} = \frac{p_t^F Y_{F,t}}{q_t^F Y_t} dt + \frac{d(q_t^F Y_t)}{q_t^F Y_t} = \mu_t^S dt + \left( \frac{(q_t^F)(s_t)}{q_t^F} \sigma_{s_t} + \bar{\sigma}_t \right) dZ_t
\]
\[
dr_t^{P_H} = \frac{d(p_t^H B_t^{H,H})}{p_H B_t^{H,H}} = (\mu_{p_t^{H,H}} + r_t^H) dt + \sigma_{p_t^{H,H}} dZ_t
\]
\[
dr_t^{P_F} = \frac{d(p_t^F B_t^{H,F})}{p_F B_t^{H,F}} = (\mu_{p_t^{H,F}} + r_t^F) dt + \sigma_{p_t^{H,F}} dZ_t
\]
And we have
\[
\sigma_{p_t^H} = -\frac{1}{\eta - 1} \sigma_{s_t}
\]
\[
\sigma_{p_t^F} = \frac{s_t}{(\eta - 1)(1 - s_t)} \sigma_{s_t}
\]
Since $\rho W^H_t = n_t Y_t$, we have

$$\frac{dW^H_t}{W^H_t} = \mu^H_t dt + \left( \frac{n'(s_t)s_t}{n_t} \sigma_{st} + \sigma_t \right) dZ_t$$

Note that

$$\sigma_t = [s_t \sigma^H, (1 - s_t) \sigma^F]$$

are linearly independent for non-degenerate $s_t$. Matching terms for $\sigma_t$, we must have that

$$\frac{\chi^H_t S^H_t}{W^H_t} + \frac{\chi^F_t S^F_t}{W^H_t} = 1$$

That is

$$p^H_t B^H_t + p^F_t B^H,F_t = 0$$

**Corollary 2.** The total capital flow of home country induced by equity trade is given by

$$dQ^H_t = S^H_t d\chi^H_t - \chi^H_t (p^H_t Y^H_{t,t}) dt + S^F_t d\chi^F_t - \chi^F_t (p^F_t Y^F_{t,t}) dt$$

$$+ d\chi^H_t dS^H_t + d\chi^F_t dS^F_t$$

And such capital flow must be financed and absorbed by trading in bonds and consumption goods

$$dQ^H_t = (\theta^B_t B^H_t + \theta^B_t B^H,F_t) W^H_t - (p^H_t C^H,H_{t,t} + p^F_t C^H,F_{t,t}) dt$$

Similarly the total capital flow of foreign country induced by equity trade is given by

$$dQ^F_t = S^H_t d\chi^F_t - \chi^F_t (p^H_t Y^H_{t,t}) dt + S^F_t d\chi^F_t - \chi^F_t (p^F_t Y^F_{t,t}) dt$$

$$+ d\chi^F_t dS^H_t + d\chi^F_t dS^F_t$$

and such capital flow must be financed and absorbed by trading in bonds and consumption goods

$$dQ^F_t = (\theta^B_t B^H_t + \theta^B_t B^H,F_t) W^F_t - (p^H_t C^F,H_{t,t} + p^F_t C^F,F_{t,t}) dt$$

**Proof.** This follows from Lemma 1. Since we have

$$W^H_t = \chi^H_t S^H_t + \chi^F_t S^F_t$$

Taking total differentiation on both sides, we have that

$$dW^H_t = \chi^H_t S^H_t d\chi^H_t + \chi^F_t S^F_t d\chi^F_t + dQ^H_t$$
Combining with dynamic budget constraint
\[
\frac{dW_t^H}{W_t^H} = \frac{\chi_t^{H,H} S_t^H}{W_t^H} \, dt_t^H + \frac{\chi_t^{H,F} S_t^F}{W_t^H} \, dt_t^F + \theta_t^{H,B_H} \, dt_t^{B_H} + \theta_t^{H,B_F} \, dt_t^{B_F} \\
- p_t^H C_{H,H,F} + p_t^F C_{H,F,F} \, dt
\]

We have that
\[
dQ_t^H = (\theta_t^{B,H,H} \, dt_t^{B_H} + \theta_t^{B,H,F} \, dt_t^{B_F}) W_t^H - (p_t^H C_{H,H,F} + p_t^F C_{H,F,F}) dt
\]

Similar proof for foreign country.

**Lemma 3.** In crisis region \( s_t \in [0, s^u] \), we have that
\[
\chi_t^{H,H} = 1, \quad \chi_t^{H,F} = \overline{\chi}^{H,F} \\
p_t^H B_t^{H,H} = p_t^F B_t^{H,F} = 0
\]

**Proof.** At the crisis region boundary \( s_t = s^u \), we have \( \chi_t^{H,H} = 1 \) and \( \chi_t^{H,F} = \overline{\chi}^{H,F} \).

\[
dQ_t^H = S_t^H d\chi_t^{H,H} - (p_t^H Y_{H,F}) dt + S_t^F d\chi_t^{H,F} - \overline{\chi}^{H,F} (p_t^F Y_{F,F}) dt + d\chi_t^{H,H} dt_s^H + d\chi_t^{H,F} dt_s^F
\]

For any realization of \( ds_t < 0 \) at \( s_t = s^u \), it must be that \( d\chi_t^{H,H} \leq 0 \) and \( d\chi_t^{H,F} \leq 0 \). To satisfy this, we must have \( d\chi_t^{H,H} \) and \( d\chi_t^{H,F} \) are deterministic for any realization of \( ds_t < 0 \) (thus, any \( s < s^u \)) at \( s_t = s^u \).

Collecting terms for \( \sigma_{s_t} \) and \( \overline{\sigma}_t \), we have
\[
dQ_t^H - \mathbb{E}[dQ_t^H] = [(S_t^H d\chi_t^{H,H} + S_t^F d\chi_t^{H,F}) \overline{\sigma}_t + (S_t^H d\chi_t^{H,H} \frac{(q_t^H)'(s_t)}{q_t^H} + S_t^F d\chi_t^{H,F} \frac{(q_t^F)'(s_t)}{q_t^F}) \sigma_{s_t})] \, dZ_t
\]

On the other side, we have
\[
dQ_t^H = (\theta_t^{B,H,H} \, dt_t^{B_H} + \theta_t^{B,H,F} \, dt_t^{B_F}) W_t^H - (p_t^H C_{H,H,F} + p_t^F C_{H,F,F}) dt
\]

which only consists of \( \sigma_{s_t} \) risk. As a result of matching terms for \( \overline{\sigma}_t \), we have
\[
S_t^H d\chi_t^{H,H} + S_t^F d\chi_t^{H,F} = 0
\]

That is, \( d\chi_t^{H,H} = d\chi_t^{H,F} = 0 \). It follows that matching terms for \( \sigma_{s_t} \) on both sides should also be 0, and we have
\[
-\theta_t^{B,H,H} + \frac{s_t}{1 - s_t} \theta_t^{B,H,F} = 0
\]

Combining with
\[
\theta_t^{B,H,H} + \theta_t^{B,H,F} = 0
\]

We have that in crisis region,
\[
\theta_t^{B,H,H} = \theta_t^{B,H,F} = 0
\]

While in crisis when \( s_t \in [0, s^U] \), \( P_t^H = (1 - \overline{\chi}^{H,F}) s_t + \overline{\chi}^{H,F} \) and \( P_t^F = (1 - \overline{\chi}^{H,F})(1 - s_t) \).

43
Because the two countries have different consumption prices which are non-degenerate stochastic processes, the same (real) bond corresponds to different return processes in the two countries. That is, real bond returns bear consumption price risks and real bonds cannot help overcome consumption price deviations. As a result, any real bond will not be traded in the constrained equilibrium. The equity holding constraints create financial friction that cannot be overcome by sovereign bonds.

**With one equity constraint** Recall the wealth of home country and its evolution

\[
W_t^H = \chi_t^H s_t^H + \chi_t^F s_t^F + p_t^H B_t^H + p_t^F B_t^F.
\]

\[
\frac{dW_t^H}{W_t^H} = \frac{\rho \chi_t^H q_t^H}{n_t} dr_t^H + \frac{\rho \chi_t^F q_t^F}{n_t} dr_t^F + \frac{p_t^H B_t^H}{W_t^H} dr_t B_t^1 + \frac{p_t^F B_t^F}{W_t^H} dr_t B_t^2 - \rho dt
\]  

(46)

Denote

\[
\frac{dW_t^H}{W_t^H} = \mu_{W_t^H} dt + \sigma_{W_t^H} dZ_t
\]  

(48)

\[
\frac{dW_t^F}{W_t^F} = \mu_{W_t^F} dt + \sigma_{W_t^F} dZ_t
\]  

(49)

\[
\frac{dn_t}{n_t} = \mu_{n_t} dt + \sigma_{n_t} dZ_t
\]  

(50)

There are two risks in this world: the aggregate consumption risk, \(\sigma_t\), and the distribution risk, \(\sigma_{st}\). Since there are four financial assets, there is some redundancy. With only one equity constraint, the two countries can still perfectly share consumption risk. So we have

\[
\sigma_{nt} = \sigma_{1-n_t} = 0
\]  

(51)

and

\[
\sigma_{W_t^H} = \sigma_{W_t^F} = \sigma_t
\]  

(52)

To find the portfolio weights on each asset, we have

\[
\sigma_t = \frac{\rho \chi_t^H q_t^H}{n_t} (\sigma_{q_t^H} + \sigma_t) + \frac{\rho \chi_t^F q_t^F}{n_t} (\sigma_{q_t^F} + \sigma_t) + \frac{p_t^H B_t^H}{W_t^H} \sigma_{p_t^H} + \frac{p_t^F B_t^F}{W_t^H} \sigma_{p_t^F}
\]  

(53)

Now we need to find \(\sigma_{q_t^H}\). In the complete market case, we have

\[
q_t^H = \mathbb{E}_t \left[ \int_0^\infty e^{-\rho \tau} s_t \, d\tau \right]
\]  

(54)

Using Ito’s lemma we have

\[
\sigma_{q_t^H} = \frac{(q_t^H)'(s_t)s_t}{q_t^H} \sigma_{s_t}
\]  

(55)

and since

\[
q_t^F = \frac{1}{\rho} - q_t^H
\]  

(56)

\[\text{The symmetric setting in discount rate and preferences matters. (conjecture)}\]
we have
\[ \sigma_{q,H} = -q_H^H \sigma_{q,F} = -\frac{(q^H)'(s_t)s_t}{q_t^F} \sigma_{st} \] (57)

And we have
\[ \sigma_{p,H} = -\frac{1}{\eta - 1} \sigma_{st} \] (58)
\[ \sigma_{p,F} = \frac{s_t}{(\eta - 1)(1 - s_t)} \sigma_{st} \] (59)

Note that
\[ \sigma_T = [s_t \sigma_1, (1 - s_t) \sigma_2] \] (60)
\[ \sigma_{\eta} = \frac{\eta - 1}{\eta} (1 - s_t)[\sigma_1, -\sigma_2] \] (61)

are linearly independent. Now coming back to the risk of home country’s wealth (53) and matching \( \sigma_t \) term, we have
\[ \rho \chi_{H,H}^t q_H^t + \rho \chi_{H,F}^t q_F^t = 1 \] (62)

That is
\[ W_t^H = \chi_{H,F}^t s_t^H + \chi_{H,F}^t S_t^F \] (63)

and thus
\[ p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = 0 \] (64)

To determine portfolio weights on bonds, we match \( \sigma_s \) terms in home country’s wealth
\[ \frac{\rho \chi_{H,H}^t q_H^t (q^H)'(s_t)_{st}}{q_t^H} + \frac{\rho \chi_{H,F}^t q_F^t}{n_t} (-\frac{(q^H)'(s_t)_{st}}{q_t^F}) + \frac{p_t^H B_t^{H,H}}{W_t^F} (-\frac{1}{\eta - 1} - \frac{s_t}{(\eta - 1)(1 - s_t)}) = 0 \] (65)

Simplified to
\[ \frac{\rho (q^H)'(s_t)_{st}(\chi_{H,H}^t - \chi_{H,F}^t)}{n_t} - \frac{p_t^H B_t^{H,H}}{W_t^F} \frac{1}{(1 - s_t)(\eta - 1)} = 0 \] (66)

So we have
\[ \frac{p_t^H B_t^{H,H}}{W_t^F} = \frac{\rho (q^H)'(s_t)_{st}(\chi_{H,H}^t - \chi_{H,F}^t)}{n_t} > 0 \] (67)

and
\[ \frac{p_t^F B_t^{H,F}}{W_t^F} = -\frac{p_t^H B_t^{H,H}}{W_t^F} < 0 \] (68)

We can also find the drift of home country wealth share \( n_t \) by looking at the drift term of the wealth. Using market clearing condition \( B_t^{H,H} = -B_t^{F,H} \) and \( B_t^{H,F} = -B_t^{F,F} \),

\[ \mu_{W_t^F} = -\frac{n_t}{1 - n_t} \mu_{nt} + \overline{\mu}_t \] (69)
\[ = \frac{E_t[dr_t^{SF}]}{dt} = -\frac{n_t}{1 - n_t} \frac{p_t^H B_t^{H,H}}{W_t^H} m_t \sigma_{\eta}^T - \rho \] (70)

and we have that
\[ \mu_{q_H} = \rho - \frac{s_t}{q_t^H} + \mu_{nt} - \sigma_{nt}^2 + \sigma_{q_H} \sigma_{nt} \] (71)
and \( \sigma_{nt} = 0 \),

\[
\mu_{q_t} = \rho - \frac{s_t}{q_t^H} + \mu_{nt}
\]  

(72)

and thus

\[
\mathbb{E}_t[dx_{t}^{S,F}] = \frac{1 - s_t}{q_t^F} + \left( - \frac{q_t^H}{q_t^F} \mu_{q_t} + \bar{\mu}_t + \sigma_{q_t^F} \sigma_t^F \right)
\]  

(73)

Substituting into (69), and solving for \( \mu_{nt} \), we have

\[
\mu_{nt} = \frac{n_t - \frac{q_t^H}{q_t^F} \sigma_{q_t^H} \sigma_t^H}{1 - n_t} = 0
\]  

(74)

as

\[
\frac{q_t^H}{q_t^F} \sigma_{q_t^H} \sigma_t^H = \frac{(q^H)'(s_{nt}) s_t (1 - s_t) (\eta - 1)}{\rho q_t^H (1 - \rho q_t^H)} \sigma_{q_t^F} \sigma_t^F = \frac{n_t \theta_t^{H,BF}}{1 - n_t} \sigma_{q_t^F} \sigma_t^F
\]  

(75)

With two equity holding constraints The second step is to explore what will happen with two equity holding constraints. Now the two countries can not always perfectly share consumption risk and the wealth shares are not always constant.

Full model special case: with two equity holding constraints \( 0 \leq \chi_{t}^{H,F} \leq \chi^{H,F} \), \( 0 \leq \chi_{t}^{F,H} \leq \chi^{F,H} \), and symmetric parameters \( \sigma_1 = \sigma_2 = \sigma \) there are three safety thresholds \( s_c, s_u \) and \( s_a, n_0 = \frac{1}{2} \):

\[
s_c = \frac{1}{2}
\]  

(76)

\[
q_1(s_u) = \frac{1 - 2\chi_{t}^{H,F}}{2\rho (1 - \chi^{H,F})}
\]  

(77)

and

\[
0 < s_a < s_u
\]  

(78)

with parameter restrictions on \((\chi^{H,F}, \eta)\). The equity shares are

\[
\chi_{1t} = 1
\]  

(79)

and

\[
\chi_{t}^{H,F} = \begin{cases} 
\chi^{H,F} & \text{if } s_t < s^U(\chi^{H,F}) \\
\frac{1 - 2\rho q_t^H}{2(1 - \rho q_t^H)} & \text{if } s^U(\chi^{H,F}) < s_t < \frac{1}{2} \\
0 & \text{if } s_t > \frac{1}{2}
\end{cases}
\]  

(80)

bond holdings are (in equilibrium)

\[
\theta_t^{H,BH} = \theta_t^{H,BF} = \theta_t^{F,BH} = \theta_t^{F,BF} = 0
\]  

(81)

where

\[
\frac{q_t^H}{n_t} = \mathbb{E}_t \left[ \int_0^\infty e^{-\rho s_z} \frac{s_z}{n_t} d\tau \right]
\]  

(82)

Proof. The equity holding constraint binds when \( s_t < s^u \). There are no cross-border bond trading.

46
The implied bond returns (within domestic country) are given as follows

\[
\frac{\mathbb{E}_t[dr_t^{B,H,H}]}{dt} - r_t^{H,f} = m_t^{T} \sigma_{p_t^H} = (\sigma_{n_t} + \sigma_t) \sigma_{p_t^H} \tag{83}
\]

\[
\frac{\mathbb{E}_t[dr_t^{B,H,F}]}{dt} - r_t^{H,f} = m_t^{T} \sigma_{p_t^F} = (\sigma_{n_t} + \sigma_t) \sigma_{p_t^F} \tag{84}
\]

\[
\frac{\mathbb{E}_t[dr_t^{B,F,H}]}{dt} - r_t^{F,f} = m_t^{T} \sigma_{p_t^H} = (- \frac{n_t}{1 - n_t}) \sigma_{n_t} + \sigma_t \sigma_{p_t^H} \tag{85}
\]

\[
\frac{\mathbb{E}_t[dr_t^{B,F,F}]}{dt} - r_t^{F,f} = m_t^{T} \sigma_{p_t^F} = (- \frac{n_t}{1 - n_t}) \sigma_{n_t} + \sigma_t \sigma_{p_t^F} \tag{86}
\]

where \(-r_t^{H,f}\) and \(-m_{1t}\) are the drift and volatility of home country’s SDF and similarly \(-r_t^{F,f}\) and \(-m_{2t}\) are for foreign country.

\[
r_t^{H,f} = \rho + \mu_{n_t} + \bar{\mu}_t + \sigma_{n_t} \bar{\sigma}_t - (\sigma_{n_t} + \bar{\sigma}_t)^2 \tag{87}
\]

\[
r_t^{F,f} = \rho - \frac{n_t}{1 - n_t} \mu_{n_t} + \bar{\mu}_t - \frac{n_t}{1 - n_t} \sigma_{n_t} \bar{\sigma}_t - (- \frac{n_t}{1 - n_t} \sigma_{n_t} + \bar{\sigma}_t)^2 \tag{88}
\]

\[
m_{1t} = \sigma_{n_t} + \bar{\sigma}_t \tag{89}
\]

\[
m_{2t} = \sigma_{1 - n_t} + \bar{\sigma}_t \tag{90}
\]

When \(s_t < s(\chi_{H,F})\), we can write out Country 1 and Country 2’s wealth as

\[
W_t^H = S_t^{H,F} + \chi_{H,F} S_t^{H,F} \tag{91}
\]

\[
W_t^F = (1 - \chi_{H,F}) S_t^{F,F} \tag{92}
\]

And from the optimization of logarithmic utility, we have

\[
\rho W_t^H = C_{H,t} = n_t \bar{Y}_t \tag{93}
\]

\[
\rho W_t^F = C_{F,t} = (1 - n_t) \bar{Y}_t \tag{94}
\]

Looking at the volatility of \(W_t^H\),

\[
\sigma_{W_t^H} = \sigma_{n_t} + \bar{\sigma}_t \tag{95}
\]

\[
= \frac{\rho q_{t}^{H,H} \sigma_{q_{t}^{H,H}} + \bar{\sigma}_t}{n_t} + \frac{\rho \chi_{H,F} q_{t}^{H,F} \sigma_{q_{t}^{H,F}}}{n_t} (\sigma_{q_{t}^{H,F}} + \bar{\sigma}_t) \tag{96}
\]

and the volatility of \(W_t^F\),

\[
\sigma_{W_t^F} = - \frac{n_t}{1 - n_t} \sigma_{n_t} + \bar{\sigma}_t \tag{97}
\]

\[
= \frac{\rho (1 - \chi_{H,F}) q_{t}^{F,F} \sigma_{q_{t}^{F,F}}}{1 - n_t} (\sigma_{q_{t}^{F,F}} + \bar{\sigma}_t) \tag{98}
\]

For both countries, portfolio weights add up to 1

\[
\frac{\rho q_{t}^{H,H}}{n_t} + \frac{\rho \chi_{H,F} q_{t}^{H,F}}{n_t} = 1 \tag{99}
\]
so we have
\[
\frac{\rho q^H_{n_t}}{n_t} + \frac{\rho \chi_{H,F} q^F_{n_t}}{n_t} + \frac{\rho (1 - \chi_{H,F}) q^F_{1 - n_t}}{1 - n_t} = 2
\] (101)

where
\[
q^H_{n_t} = \mathbb{E}_t \left[ \int_0^\infty e^{-\rho t} \frac{s_\tau}{n_\tau} d\tau \right]
\] (102)
\[
q^F_{n_t} = \mathbb{E}_t \left[ \int_0^\infty e^{-\rho t} \frac{1 - s_\tau}{n_\tau} d\tau \right]
\] (103)
\[
q^F_{1 - n_t} = \mathbb{E}_t \left[ \int_0^\infty e^{-\rho t} \frac{1 - s_\tau}{1 - n_\tau} d\tau \right]
\] (104)

That is,
\[
\mathbb{E}_t \left[ \int_0^\infty e^{-\rho t} \left( \frac{s_\tau + \chi_{H,F}(1 - s_\tau)}{n_\tau} + \frac{(1 - \chi_{H,F})(1 - s_\tau)}{1 - n_\tau} \right) d\tau \right] = \frac{2}{\rho}
\] (105)

Using Feynman-Kac formula, we have
\[
\frac{s_t + \chi_{H,F}(1 - s_t)}{n_t} + \frac{(1 - \chi_{H,F})(1 - s_t)}{1 - n_t} = 2
\] (106)

So \( n_t \) is a function of the only state variable \( s_t \) (the other solution \( n_t = \frac{1}{2} \) is not achievable with equity constraint binding).
\[
n_t = (1 - \chi_{H,F}) s_t + \chi_{H,F}
\] (107)
\[
1 - n_t = (1 - \chi_{H,F})(1 - s_t)
\] (108)
\[
\theta_t^{H,B_H} = \theta_t^{H,B_F} = 0
\] (109)

And also we have
\[
\frac{C_{H,t}}{C_{F,t}} = \frac{n_t}{1 - n_t}
\] (110)

Now we can solve for prices of risks,
\[
m_{1t} = \sigma_{n_t} + \sigma_t = \frac{(1 - \chi_{H,F}) s_t}{(1 - \chi_{H,F}) s_t + \chi_{H,F} \sigma_{s_t}} + \sigma_t
\] (111)

and solve for \( s^a \) using
\[
\mathbb{E}_t[dr_t^{H,H} - dr_t^{B_H}] = m_{1t}(\sigma_{p_H} - \sigma_{p_F}) < 0
\] (112)

we have
\[
\frac{2(1 - \overline{\chi}_{H,F}) s^2 + (2\overline{\chi}_{H,F} \eta + (1 - \overline{\chi}_{H,F})(\eta - 2))s - \eta \overline{\chi}_{H,F}}{(1 - \overline{\chi}_{H,F}) s_t + \overline{\chi}_{H,F}} > 0
\] (113)

For \( 0 < s^a < s^u \), we need the right range for parameter pair \((\overline{\chi}_{H,F}, \eta)\). For example, if \( \overline{\chi}_{H,F} = 0 \), we have \( s^u = \frac{1}{2} \) and \( s^a = \frac{2 - \eta}{2} \). And if \( \eta = \infty \), we have \( s^a = \frac{\overline{\chi}_{H,F}}{1 + \overline{\chi}_{H,F}} \).

With a special parameter case, \( \rho = \left( \frac{\eta - 1}{\eta} \sigma \right)^2 \), we can solve everything we need analytically. The
price of equity 1 in unconstrained case is

\[ q_t^H(s) = \frac{1}{2\rho} \left( 1 + \frac{1-s}{s} \ln(1-s) - \frac{s}{1-s} \ln(s) \right) \]  (114)

\[ q_t'(s) = -\frac{1}{2\rho} \frac{1}{s(1-s)} \left( 1 + \frac{1-s}{s} \ln(1-s) + \frac{s}{1-s} \ln(s) \right) \]  (115)

\[ q_t''(s) = -\frac{1}{2\rho} \frac{1}{s^2(1-s)^2} \left( (2s - 1) - \frac{(1-s)^2}{s} \ln(1-s) + \frac{s^2}{1-s} \ln(s) \right) \]  (116)
B Additional Graphs and Tables

Table B.1: Annualized UIP Premium for G10 Currencies

Note: This panel of figures present the pattern of UIP premiums for G-10 currencies over the 2000 - 2021 sample. The UIP premiums are defined as the annualized excess returns of local-currency one-year government bond yields against the synthetic USD yields and are in log points. The grey area corresponds to the months of the Great Financial Crisis in 2008.
Table B.2: Annualized UIP Premium for EME currencies

Note: This panel of figures present the pattern of UIP premiums for emerging market economies over the 2000 - 2021 sample. The UIP premiums are defined as the annualized excess returns of local-currency one-year government bond yields against the synthetic USD yields and are in log points. The grey area corresponds to the months of the Great Financial Crisis in 2008.
Table B.3: G10-currency annualized UIP on relative country size

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Standard errors in parentheses
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: This table presents the OLS results for monthly UIP deviations on the relative size of the country to the “world GDP” (defined as sum of the country and the U.S.). The UIP deviations are annualized and defined in excess returns in local currency against the USD and measured in log points. The relative country size of each country is computed as its GDP over the sum with the U.S. GDP. Column (1) covers the full sample at monthly frequency from 2000 to 2021; column (2) uses the sub-sample before the 2008 financial crisis and column (3) uses the post-crisis sub-sample. All regressions include year and country fixed effects.

Table B.4: EME-currency annualized UIP on the relative country size

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Full Sample</td>
<td>Year &lt; 2008</td>
<td>Year &gt; 2008</td>
</tr>
<tr>
<td>Share of GDP over World</td>
<td>0.480**</td>
<td>-2.225**</td>
<td>-0.114</td>
</tr>
<tr>
<td></td>
<td>(0.219)</td>
<td>(1.057)</td>
<td>(0.316)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.000862</td>
<td>0.132***</td>
<td>0.0102</td>
</tr>
<tr>
<td></td>
<td>(0.00919)</td>
<td>(0.0322)</td>
<td>(0.0142)</td>
</tr>
<tr>
<td>Country + Year FE</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Observations</td>
<td>1640</td>
<td>416</td>
<td>1224</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.2922</td>
<td>0.2638</td>
<td>0.2921</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.280</td>
<td>0.236</td>
<td>0.280</td>
</tr>
</tbody>
</table>

Standard errors in parentheses
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: This table presents the OLS results for monthly UIP deviations on the relative size of the country to the “world GDP” (defined as sum of the country and the U.S.). The UIP deviations are annualized and defined in excess returns in local currency against the USD and measured in log points. The relative country size of each country is computed as its GDP over the sum with the U.S. GDP. Column (1) covers the full sample at monthly frequency from 2000 to 2021; column (2) uses the sub-sample before the 2008 financial crisis and column (3) uses the post-crisis sub-sample. All regressions include year and country fixed effects.
Table B.5: G10-currency annualized UIP on within-G10-group country size

<table>
<thead>
<tr>
<th></th>
<th>(1) Full Sample</th>
<th>(2) Year &lt; 2008</th>
<th>(3) Year &gt; 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within-G10 GDP Share (%)</td>
<td>-1.072***</td>
<td>-0.665***</td>
<td>-2.195***</td>
</tr>
<tr>
<td></td>
<td>(0.108)</td>
<td>(0.156)</td>
<td>(0.267)</td>
</tr>
<tr>
<td>Constant</td>
<td>11.88***</td>
<td>11.32***</td>
<td>20.95***</td>
</tr>
<tr>
<td></td>
<td>(1.094)</td>
<td>(1.582)</td>
<td>(2.679)</td>
</tr>
<tr>
<td>Country + Year FE</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Observations</td>
<td>2508</td>
<td>960</td>
<td>1548</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.4408</td>
<td>0.4632</td>
<td>0.3796</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.434</td>
<td>0.453</td>
<td>0.371</td>
</tr>
</tbody>
</table>

Standard errors in parentheses
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: This table presents the OLS results for monthly UIP deviations on the relative size of the country (measured by nominal GDP) within the G-10 currency group. The UIP deviations are annualized and defined in excess returns in local currency against the USD and measured in log points. The relative country size of each country is in percentage points and computed as its share over the total nominal GDP of all G10 countries (other than the US). Column (1) covers the full sample at monthly frequency from 2000 to 2021; column (2) uses the sub-sample before the 2008 financial crisis and column (3) uses the post-crisis sub-sample. All regressions include year and country fixed effects.

Table B.6: EME-currency annualized UIP on within-EME-group country size

<table>
<thead>
<tr>
<th></th>
<th>(1) Full Sample</th>
<th>(2) Year &lt; 2008</th>
<th>(3) Year &gt; 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within-EME GDP Share (%)</td>
<td>0.0732</td>
<td>-0.598</td>
<td>0.0400</td>
</tr>
<tr>
<td></td>
<td>(0.0865)</td>
<td>(0.377)</td>
<td>(0.117)</td>
</tr>
<tr>
<td>Constant</td>
<td>1.374*</td>
<td>13.18***</td>
<td>0.0162</td>
</tr>
<tr>
<td></td>
<td>(0.832)</td>
<td>(3.718)</td>
<td>(1.096)</td>
</tr>
<tr>
<td>Country + Year FE</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Observations</td>
<td>2206</td>
<td>533</td>
<td>1673</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.2977</td>
<td>0.2334</td>
<td>0.2906</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.288</td>
<td>0.207</td>
<td>0.281</td>
</tr>
</tbody>
</table>

Standard errors in parentheses
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: This table presents the OLS results for monthly UIP deviations on the relative size of the country (measured by nominal GDP) in the EME group. The UIP deviations are annualized and defined in excess returns in local currency against the USD and measured in log points. The relative country size of each country is in percentage points and computed as its share over the total nominal GDP of all EME countries. Column (1) covers the full sample at monthly frequency from 2000 to 2021; column (2) uses the sub-sample before the 2008 financial crisis and column (3) uses the post-crisis sub-sample. All columns include year and country fixed effects in the OLS.
Table B.7: Relative G10 (or EME) Country Size

**Note:** This panel of figures present the ratio of nominal GDP of a country over the sum of GDP of the country and of the U.S. The figure for G10 currency countries are on the left and emerging market economies are on the right. Each line represents the country-specific GDP ratio. The bolded line is the weighted average ratio for the G10 group (left) and the EME group (right).

Table B.8: Average UIP Premium and Country Size Share within the G10 (or EME) Group

**Note:** This panel of figures present the scatter plot of average UIP premium in our sample against the average country size share (measured by nominal GDP) for each currency in their respective G10 currency group (left) or EME group (right). UIP deviations are in log points and are annualized and averaged across time for the sample period of 2000 to 2021. Each dot represent a currency labeled by its currency name.